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ON CENTRAL EXTENSIONS AND DEFINABLY COMPACT GROUPS IN O-MINIMAL STRUCTURES

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Abstract. We prove several structural results on definably compact groups G in o-minimal expansions of real closed fields such as (i) G is definably an almost direct product of a semisimple group and a commutative group, (ii) $(G,\cdot)$ is elementarily equivalent to $(G/G^{00},\cdot)$. We also prove results on the internality of finite covers of G in an o-minimal environment, as well as deducing the full compact domination conjecture for definably compact groups from the semisimple and commutative cases which were already settled.

These results depend on key theorems about the interpretability of central and finite extensions of definable groups, in the o-minimal context. These methods and others also yield interpretability results for universal covers of arbitrary definable real Lie groups.

1. Introduction and preliminaries

This paper is motivated partly by questions coming out of our paper [13], especially whether, for a definably compact group G in an o-minimal structure (say expanding a real closed field), G and G/G^{00} are elementarily equivalent as groups. We solve this problem (see Theorem 7.1) and in the process manage to tie up several loose ends regarding definable groups in o-minimal structures. For now we will just say "o-minimal structure" $\mathcal{M}$ but often there are additional assumptions on $\mathcal{M}$ such as expanding a real closed field, or expanding an ordered group, which appear explicitly in the statements. One of the main results, Theorem 6.1, considers a definably connected central extension $\tilde{G}$ of a semisimple group G by A, all definable in $\mathcal{M}$ and says that the exact sequence $A \to \tilde{G} \to G$ of groups, is essentially bi-interpretable with the pair $\langle G, A\rangle$ of groups. Corollary 6.2 deduces that any such $\tilde{G}$ (in particular any definably compact group) is elementarily equivalent, as a group, to a semialgebraic real Lie group. From this it is not hard to deduce (Corollary 6.4) that a definably compact definably connected group is definably an almost direct product of a semisimple group and a commutative group. Corollary 6.5 strengthens this to central extensions of definably compact semisimple groups. Section 8 contains interpretability and internality results for finite (not necessarily central) extensions of groups, again definable in an o-minimal structure $\mathcal{M}$. In section 9, definable groups which are not necessarily definably connected are considered, and Corollary 6.4 (elementary equivalence to semialgebraic Lie groups) is generalized (see Theorem 9.4). In section 10 we point out how the compact domination conjecture (for

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definably compact groups in o-minimal expansions of real closed fields) follows from our results, together with earlier work.

The above results rely on the main technical theorem (Theorem 2.1) about the interpretability of central extensions in an o-minimal context, which appears in section 2. The theorem roughly says that under certain assumptions, a definable central extension $\tilde{G}$ of a definable group G can be interpreted in the two-sorted structure $\langle G, Z(\tilde{G})\rangle$ (possibly, after expanding G by definably connected components of some definable sets). In fact we also note that if the base o-minimal structure $\mathcal{M}$ is an expansion of the reals (in which case we sometimes call a group definable in $\mathcal{M}$, a definable real Lie group), the assumption that $\tilde{G}$ is definable can be omitted, obtaining interpretability results for central topological extensions of suitable definable Lie groups (see section 2.1 and Theorem 2.8). A version for finite extensions appears in section 8.1 (see Theorem 8.4). Also in section 8.1 a result with a similar flavour is proved for arbitrary connected definable real Lie groups G: for example, the universal cover $\pi : \tilde{G} \to G$ is interpretable in the two sorted structure consisting of the given o-minimal expansion of $\mathbb{R}$ together with $\langle ker(\pi), +\rangle$.

Sections 3, 4 and 5 are devoted to checking that various hypotheses of Theorem 2.1 hold in the cases we are interested in.

Our notation is in on the whole standard. However, as we are concerned with issues of interpretability in certain reducts, we will mention the relevant notation.

We do not in general keep track of the definable-interpretable distinction. Hence, when we talk about *definable* sets, groups etc. in a structure $\mathcal{N}$, we mean definable (with parameters) in $\mathcal{N}^{eq}$. We sometimes say (by abuse of notation) that a set X is $\mathcal{N}$-definable if X is definable in the above sense (with parameters) in the structure $\mathcal{N}$.

In general, $\mathcal{M}$ is an o-minimal structure, with M is its universe and as a rule our groups G, $\tilde{G}$, H etc. are all definable in $\mathcal{M}$ (again with parameters). However, in order to use results from [23] about topology of groups we add the extra assumption that *in the structure $\mathcal{M}$, every such group is definably isomorphic to a group whose universe is a subset of M^n*. All main results assume that $\mathcal{M}$ has definable Skolem functions so this assumption is obtained for free in that setting. We may often want to view the group G as a structure in just the group language, i.e. $\langle G, \cdot\rangle$, in which case we sometimes write (with maybe some ambiguity) this structure as just G.

We now review some earlier results, mainly about definably simple and definable semisimple groups.

A *definably simple* group is a definable, non-abelian group with no definable normal subgroup. A *semisimple* group is a definable group with no infinite definable normal abelian subgroup (because of DCC, the definability requirement is superfluous).

We summarize the main results which we will be using here:

Fact 1.1. *Let $\mathcal{M}$ be an o-minimal structure.*

(1) *If G is a definably simple group then there is in $\mathcal{M}$ a definable real closed field R and a real algebraic group H defined over $R_{alg} \subseteq R$ the subfield of real algebraic numbers, such that G is definably isomorphic in $\mathcal{M}$ to $H(R)^0$, the definably connected component of $H(R)$ (see* [18, 4.1] *for the existence of*

an algebraic group H and [20, 5.1] *and its proof, for the fact that H can be defined over R_{alg}).*

(2) *If G is definably simple then it is either bi-interpretable, over parameters, with a real closed field or with an algebraically closed field of characteristic zero,* [19].

(3) *If G is definably connected and semisimple then $Z(G)$ is finite and $G/Z(G)$ is definably isomorphic in $\mathcal{M}$ to the direct product of definably simple groups (see* [18, 4.1]*).*

(4) *If G is definably compact and definably connected then either G is abelian or $G/Z(G)$ is semisimple,* [21, 5.4].

(5) *If G is definably simple in a sufficiently saturated structure then it is abstractly simple if and only if G is not definably compact,* [20, 6.3].

Note that the bi-interpretability of (2) above is necessarily over parameters (see [19, Remark 4.11]). Also, the Skolem functions assumption can be omitted if G is definable in the real sort of $\mathcal{M}$.

2. The main interpretability theorem

We recall that $\mathcal{M}$ is an *o*-minimal structure.

Let G be a definable group. By a *definable central extension of G* we mean the following data: definable groups $A, \tilde{G}$, definable homomorphisms: $i : A \to \tilde{G}$, $\pi : \tilde{G} \to G$ with

$$1 \to A \xrightarrow{i} \tilde{G} \xrightarrow{\pi} G \to 1$$

exact and $i(A)$ central in $\tilde{G}$. We let $\langle A, \tilde{G}, G, i, \pi \rangle$ denote the three group structures, together with the maps i and π. We say that the central sequence above is (definably) isomorphic to another definable exact sequence

$$1 \to A_1 \xrightarrow{i_1} \tilde{G}_1 \xrightarrow{\pi_1} G_1 \to 1$$

if there are (definable) group isomorphisms $h_A : A_1 \to A$, $h_{\tilde{G}} : \tilde{G}_1 \to \tilde{G}$ and $h_G : G_1 \to G$, which commute with the exact sequence maps.

When G is a definable group whose universe is in the real sort of an an o-minimal $\mathcal{M}$ then it has a canonical group topology (see [23]), with respect to which every $\mathcal{M}$-definable subset of G has finitely many definably connected components. Let $\mathbf{G}$ be an arbitrary expansion of the group G (not necessarily definable in $\mathcal{M}$). We say that $\mathbf{G}$ *has property ρ* if for every $\langle G, \cdot \rangle$-definable $X \subseteq G^n$, every definably connected component of X (with respect to the group topology of G) is definable in $\mathbf{G}$, possibly over new parameters.

Given the abelian group A, we use $\langle \mathbf{G}, A \rangle$ to denote the two-sorted structure of the two groups, where G is equipped with its $\mathbf{G}$-structure and A with just its group structure.

Recall that for a definable group H, the commutator subgroup $[H, H]$ is a countable union of definable sets, which might not be definable itself. More precisely, if we denote by $[H, H]_n$ the definable set of all products of n commutators in H, then $[H, H] = \bigcup_{n \in \mathbb{N}} [H, H]_n$.

Theorem 2.1. *Let $\mathcal{M}$ be an o-minimal structure and assume $E = 1 \to A \xrightarrow{i} \tilde{G} \xrightarrow{\pi} G \to 1$ is an $\mathcal{M}$-definable central extension of G, $\mathbf{G}$ an arbitrary expansion of G such that:*

(1) $\mathbf{G}$ has property ρ.
(2) For every n, the set $i(A) \cap [\tilde{G}, \tilde{G}]_n$ is finite (we call this property Z*).*
(3) There exists $r \in \mathbb{N}$ with $G = [G, G]_r$.

Then E (that is the structure $\langle A, \tilde{G}, G, i, \pi \rangle$) can be defined in $\langle \mathbf{G}, A \rangle$ over A and G. More precisely, there is an exact sequence

$$E' = 1 \to A \to \tilde{G}' \to G \to 1,$$

definable in $\langle \mathbf{G}, A \rangle$ over an imaginary parameter $\bar{c}$, such that E' is definably isomorphic, in the structure $\langle A, \tilde{G}, \mathbf{G}, i, \pi \rangle$) (note that G is expanded), to the sequence E, with h_A, h_G the identity maps.

The imaginary parameter $\bar{c}$ names a map from the set of definably connected components of a 0-definable set in the group $\langle G, \cdot \rangle$, onto a finite subset of A. This map is 0-definable in the structure $\langle A, \tilde{G}, G, i, \pi \rangle$.

Proof. Note that our assumption implies that $\tilde{G}$ can be written as the group product of the subgroups $i(A)$ and $[\tilde{G}, \tilde{G}]_r$. Our goal is to produce an $\mathcal{M}$-definable surjective map $j : G^{2r} \times A \to \tilde{G}$, such that the pull-back under j of equality in $\tilde{G}$ and of the group operation are both definable in $\langle \mathbf{G}, A \rangle$.

For $x, y \in G$, we let $[x, y] = xyx^{-1}y^{-1}$. For $n \geqslant 0$, we let $w_n(x_1, y_1, \ldots, x_n, y_n)$ be the word in the free group given by the product of the n-commutators $[x_1, y_1] \cdots [x_n, y_n]$. For any group K, we let $F_{n,K} : K^{2n} \to K$ be the associated function which evaluates the word w_n in K. The image of K^{2n} under $F_{n,K}$ is exactly $[K, K]_n$. We also have, for $\bar{h}_1 \in K^{2m}, \bar{h}_2 \in K^{2n}$,

$$F_{m,K}(\bar{h}_1) \cdot F_{n,K}(\bar{h}_2) = F_{m+n,K}(\bar{h}_1, \bar{h}_2),$$

and if for $\bar{h} = (x_1, y_1, \ldots, x_n, y_n)$ we let $inv(\bar{h}) = (y_n, x_n, y_{n-1}, x_{n-1}, \ldots, y_1, x_1)$ then

$$F_{n,K}(\bar{h})^{-1} = F_{n,K}(inv(\bar{h})).$$

We use π to denote the map from $\tilde{G}^{2n}$ to G^{2n} which is induced by π in each coordinate.

Claim 2.2. *For every $\bar{g}_1, \bar{g}_2 \in \tilde{G}^{2n}$, if $\pi(\bar{g}_1) = \pi(\bar{g}_2)$ then $F_{n,\tilde{G}}(\bar{g}_1) = F_{n,\tilde{G}}(\bar{g}_2)$.*

Proof. This is immediate from the fact that each coordinate of $\bar{g}_1$ differs from the corresponding coordinate of $\bar{g}_2$ by a central element of $i(A)$, and on tuples from the center of $\tilde{G}$, the map $F_{n,\tilde{G}} = e$. □

It follows from 2.2 that there is an $\mathcal{M}$-definable surjective map $k_n : G^{2n} \to [\tilde{G}, \tilde{G}]_n$ such that $F_{n,\tilde{G}}$ factors through π and k_n (see diagram below) . Also, for $\bar{g}_1 \in G^{2m}, \bar{g}_2 \in G^{2n}$, we have

$$k_m(\bar{g}_1) \cdot k_n(\bar{g}_2) = k_{m+n}(\bar{g}_1, \bar{g}_2) \text{ and } k_n(\bar{g}_1)^{-1} = k_n(inv(\bar{g})).$$

$$\begin{tikzcd}
\tilde{G}^{2n} \arrow[r, "F_{n,\tilde{G}}"] \arrow[d, "\pi"'] & \tilde{G} \arrow[d, "\pi"] \\
G^{2n} \arrow[ur, "k_n"] \arrow[r, "F_{n,G}"'] & G
\end{tikzcd}$$

Fact 2.3. *(i) The function $k_n : G^{2n} \to \tilde{G}$ is continuous (here and below we always refer to the group topology).*

(ii) For every $\bar{g}_1, \bar{g}_2 \in G^{2n}$, the following are equivalent:

(1) *$F_{n,G}(\bar{g}_1) = F_{n,G}(\bar{g}_2)$.*
(2) *$k_n(\bar{g}_1)$ and $k_n(\bar{g}_2)$ are in the same $i(A)$-coset in $\tilde{G}$.*
(3) *$F_{2n,G}(\bar{g}_1, inv(\bar{g}_2)) = e$.*

(iii) For every $\bar{g} \in G^{2n}$, $F_{n,G}(\bar{g}) = e$ if and only if $k_n(\bar{g}) \in i(A)$.

Proof. (i) Because the group topology on G equals the quotient topology (see [1, Theorem 4.3]) and because $F_{n,\tilde{G}}$ is continuous, the map k_n is also continuous.

(ii) (1) $\Leftrightarrow$ (2): Because π is a homomorphism, we have $\pi k_n(\bar{g}) = F_{n,G}(\bar{g})$, for every $\bar{g} \in \tilde{G}^{2n}$. Hence, $F_{n,G}(\bar{g}_1) = F_{n,G}(\bar{g}_2)$ if and only if $\pi k_n(\bar{g}_1) = \pi k_n(\bar{g}_2)$ if and only if $k_n(\bar{g}_1)$ and $k_n(\bar{g}_2)$ are in the same $i(A)$-coset.

(2) $\Leftrightarrow$ (3): $k_n(\bar{g}_1) \cdot k_n(\bar{g}_2)^{-1} \in i(A)$ if and only if $k_{2n}(\bar{g}_1, inv(\bar{g}_2)) \in i(A)$ if and only if $F_{2n,G}(\bar{g}_1, inv(\bar{g}_2)) = e$.

(iii) is immediate from (ii), by taking $\bar{g}_2 = (e, \dots, e)$. □

For $k \geqslant 0$, we let
$$G(k) = \{\bar{g} \in G^{2k} : F_{k,G}(\bar{g}) = e\}.$$

By our assumption, the set $i(A) \cap [\tilde{G}, \tilde{G}]_n$ is finite for every n. Because k_n is continuous and surjective on $[\tilde{G}, \tilde{G}]_n$ (since $F_{n,\tilde{G}}$ is) we have:

Fact 2.4. *The set $k_n(G(n))$ equals $i(A) \cap [\tilde{G}, \tilde{G}]_n$, and the function k_n is constant on every definably connected component of $G(n)$.*

Given n, let $b_1, \dots, b_{\ell_n} \in A$ be such that
$$\{i(b_1), \dots, i(b_{\ell_n})\} = i(A) \cap [\tilde{G}, \tilde{G}]_n.$$
We have a corresponding partition of $G(n)$ into relatively clopen sets $W_n(b_1), \dots, W_n(b_{\ell_n})$, with $k_n(W_n(b_j)) = \{i(b_j)\}$. Each $W_n(b_j)$ is a finite union of components of the set $G(n)$ which is itself $\langle G, \cdot \rangle$-definable. Hence, by property ρ, each such $W_n(b_j)$ is **G**-definable, possibly over some parameters.

The interpretation

We fix r such that $\tilde{G} = i(A) \cdot [\tilde{G}, \tilde{G}]_r$.

The Universe: Consider the map $j : A \times G^{2r} \to \tilde{G}$ defined by
$$j(a, \bar{g}) = i(a) \cdot k_r(\bar{g}).$$

Because k_r is surjective on $[\tilde{G}, \tilde{G}]_r$, the map j is surjective on $\tilde{G}$, and we have $j(a_1, \bar{g}_1) = j(a_2, \bar{g}_2)$ if and only if $k_r(\bar{g}_1) \cdot k_r(\bar{g}_2)^{-1} = i(a_1)^{-1} \cdot i(a_2)$ if and only if
$$k_{2r}(\bar{g}_1, inv(\bar{g}_2)) = i(a_1^{-1} \cdot a_2).$$

Let
$$B_{2r} = \{b_1, \dots, b_{\ell_{2r}}\} \subseteq A$$
be such that $i(B_{2r}) = i(A) \cap [\tilde{G}, \tilde{G}]_{2r}$, and let
$$\mathcal{W}_{2r} = \{W_{2r}(b_1), \dots, W_{2r}(b_{\ell_{2r}})\}$$
be the corresponding partition of $G(2r)$, as given above (namely, $k_{2r}(W(b_j)) = i(b_j)$). Let
$$c_{2r} : B_{2r} \to \mathcal{W}_{2r}$$
be the bijection which sends b_j to $W(b_j)$. Note that the map c_{2r} is 0-definable in the structure $\langle A, \tilde{G}, G, i, \pi \rangle$ because $W(2r)$, B_{2r} and k_{nr} are 0-definable there.

Consider now the equivalence relation $\sim$ induced on $A \times G^{2r}$ by the map j. It is defined by

$$j(a_1, \bar{g}_1) = j(a_2, \bar{g}_2) \Leftrightarrow (\bar{g}_1, inv(\bar{g}_2)) \in c_{2r}(a_1^{-1}a_2).$$

We let $\mathcal{U} = A \times G^{2r}/\sim$. Notice that the equivalence relation is 0-definable in the pure group language of A, the pure group language of G, together with a function symbol for $c_{2r} : B_{2r} \to \mathcal{W}_{2r}$, which in particular names the finite set $B_{2r} \subseteq A$. By property ρ, the function c_{2r} itself is definable, over parameters, in $\langle \mathbf{G}, A \rangle$ (one way to obtain c_{2r} is by naming each element of B_{2r} and then naming an element in each $W(b_j) \in \mathcal{W}_{2r}$).

We denote by $\lfloor (a, \bar{g}) \rfloor$ the $\sim$-class of $(a, \bar{g})$.

The group operation: We now consider the pull-back on $\mathcal{U}$, via the map j, of the group operation from $\tilde{G}$: We get (because $i(A)$ is central) for every $(b, \bar{h}), (a_1, \bar{g}_1), (a_2, \bar{g}_2) \in A \times G^{2r}$,

$$\lfloor (b, \bar{h}) \rfloor = \lfloor (a_1, \bar{g}_1) \rfloor \cdot \lfloor (a_2, \bar{g}_2) \rfloor \Leftrightarrow b \cdot k_r(\bar{h}) = a_1 \cdot a_2 \cdot k_r(\bar{g}_1) \cdot k_r(\bar{g}_2)$$

if and only if

$$(\bar{g}_1, \bar{g}_2, inv(\bar{h})) \in c_{3r}(a_2^{-1}a_1^{-1}).$$

As before, this last expression can be defined using the pure group structure of A and G, and a function symbol for $c_{3r} : B_{3r} \to \mathcal{W}_{3r}$. The map c_{3r} itself is defined, over parameters, in $\langle \mathbf{G}, A \rangle$.

We thus proved the interpretation of the group $\tilde{G}$ in $\langle \mathbf{G}, A \rangle$, over the imaginary parameter $\bar{c} = (c_{2r}, c_{3r})$. The map $i : A \to \tilde{G}$ is interpreted by $i(a) = \lfloor (a, (e, \cdots, e)) \rfloor$ and the map $\pi : \tilde{G} \to G$ is interpreted via $\pi(\lfloor (a, \bar{g}) \rfloor) = F_{r,G}(\bar{g})$.

We therefore obtain in $\langle \mathbf{G}, A \rangle$ a central extension of G which is isomorphic to the original one, as required. $\square$

Corollary 2.5. *Consider the assumptions of Theorem 2.1 and assume further that* $\mathbf{G}$ *is just the group structure* G*. Then* $\langle A, \tilde{G}, G, i, \pi \rangle$ *and* $\langle G, A \rangle$ *are bi-interpretable over parameters. The parameters are in* G *and* A *and they are* 0*-definable in* $\langle A, \tilde{G}, G, i, \pi \rangle$.

Proof. This is immediate from Theorem 2.1 (using the fact that the isomorphism between the two exact sequences is the identity when restricted to A and to G, in the notation of that theorem). $\square$

Remark 2.6. *Let us return to the the imaginary parameters* c_{2r}, c_{3r} *used in Theorem 2.1: The maps* c_{2r}, c_{3r} *define correspondences between finite subsets* $B_{2r}, B_{3r} \subseteq A$*, respectively, and definably connected components of* 0*-definable sets in* $\langle G, \cdot \rangle$*. However, in an o-minimal structure the definably connected components of a* 0*-definable set are themselves* 0*-definable and therefore* c_{2r} *and* c_{3r} *are* 0*-definable in the structure*

$$\langle M, <, \langle G, \cdot \rangle, \langle A, \cdot, B_{2r}, B_{3r} \rangle \rangle$$

(by that we mean that we add predicates for B_{2r} *and* B_{3r}*). In the special case that* A *itself is finite the sets* B_{2r} *and* B_{3r} *are* 0*-definable in* $\langle M, <, \langle A, \cdot \rangle \rangle$*, so these predicates can be omitted. We will later make use of this fact.*

Finally let us mention an easy general result on definable splitting, which we do not really use, but nevertheless is in the spirit of the other results.

Fact 2.7. *Suppose that $E : A \to \tilde{G} \to G$ is a central extension (of abstract) groups, and that $G = [G,G]_k$ for some k. Suppose that E splits abstractly, then it splits definably (in the structure $\langle A, \tilde{G}, G, i, \pi \rangle$).*

Proof. By the splitting assumption, $\tilde{G}$ can be written abstractly as a direct product of $i(A)$ and a subgroup $H \subseteq \tilde{G}$, with $\pi : H \to G$ an isomorphism. It follows that $[\tilde{G}, \tilde{G}] \subseteq H$ and because $[\tilde{G}, \tilde{G}]_k$ projects onto G we have $[\tilde{G}, \tilde{G}]_k = [G, G] = H$. In particular $H = [\tilde{G}, \tilde{G}]$ is definable hence $\tilde{G}$ split definably. $\square$

2.1. **The real case.** There was actually not much use of o-minimality in the proof of Theorem 2.1. Mainly, it was used in order to obtain the canonical partition of $G(w_{2n})$ into finitely many definably connected components, on each of which the map k_{2n} is constant. Because of o-minimality this partition could be read-off just using G (and the definably connected components of G-definable sets), independently of $\tilde{G}$ and π. In particular, if we work over the reals then this assumptions can be partially omitted:

We say that

$$E : 1 \to A \xrightarrow{i} \tilde{G} \xrightarrow{\pi} G \to 1$$

is a topological central extension of G, if $A, \tilde{G}$ and G are topological groups, and the maps i and π are homomorphisms of topological groups. When G is definable in an o-minimal structure (but possibly not $\tilde{G}$ and A), we always consider G with its o-minimal topology.

Theorem 2.8. *Let $\mathcal{M}$ be an o-minimal structure over the real numbers, G a definable group in $\mathcal{M}$. Let $E = 1 \to A \xrightarrow{i} \tilde{G} \xrightarrow{\pi} G \to 1$ be a topological central extension of G (so A, $\tilde{G}$, π and i are not assumed to be definable), $\mathbf{G}$ an arbitrary expansion of G such that:*
(1) $\mathbf{G}$ has property ρ.
(2) For every n, the set $i(A) \cap [\tilde{G}, \tilde{G}]_n$ is finite.
(3) There exists $r \in \mathbb{N}$ with $G = [G,G]_r$.

Then there is an exact sequence $E' : 1 \to A \to \tilde{G}' \to G \to 1$, definable in the structure $\langle \mathbf{G}, A \rangle$ (over parameters) such that E' is isomorphic (in the group language) to the sequence E, with h_A, h_G the identity maps.

If, moreover, we assume that A is definable in $\mathcal{M}$ (so the group $\tilde{G}'$ is endowed with the o-minimal topology) and $\mathcal{M}$ then the isomorphism between $\tilde{G}$ and $\tilde{G}'$ is also a topological one.

Later on, in section 8, we will show how, for finite central extensions, assumption (3) can be omitted. For now, let's note that the above already implies that every finite topological cover of $SL(2, \mathbb{R})$ is topologically isomorphic to a semialgebraic cover.

Proof of the theorem All one has to do is go back to the proof of Theorem 2.1 and see where definability and o-minimality was used. While the existence of the map $k_n : G^{2n} \to \tilde{G}$ is just a group theoretic fact (of course now k_n is not definable in M), something should be said about the continuity of k_n, mentioned in 2.3. Indeed, this just follows from the fact that the continuous homomorphism $\pi : \tilde{G} \to G$ is a quotient map. The rest of 2.3 is just group theoretic.

Proceeding to 2.4, we still have $k_n(G(n)) = i(A) \cap [\tilde{G}, \tilde{G}]_n$ and because of our assumption, this set is finite, which implies by continuity, that k_n is locally constant.

We now use the fact that over the reals the definably connected components are just the usual connected components and conclude that k_n is constant on every (definably) connected component of the $\mathbf{G}$-definable set $G(n)$. Since the rest of the argument takes place fully in $\langle \mathbf{G}, A\rangle$, we obtain in this last structure a definable central exact sequence

$$E : 1 \to A \xrightarrow{i'} \tilde{G}' \xrightarrow{\pi'} G \to 1,$$

together with a group isomorphism $h_{\tilde{G}} : \tilde{G}' \to \tilde{G}$, such that all maps commute (with the identity maps on A and G).

Assume now that A itself is definable in the o-minimal structure $\mathcal{M}$. Let's see why $h_{\tilde{G}}$ is a topological homeomorphism as well.

The group $\tilde{G}'$ is obtained as a quotient of $A \times G^{2r}$ by an $\mathcal{M}$-definable equivalence relation $\sim$ which is itself the pre-image of equality under the continuous map j. The isomorphism $h_{\tilde{G}} : \tilde{G}' \to G$ is just the map induced by j. Note that by [5], the structure $\mathcal{M}$ has definable choice functions for subsets of $A \times G$, hence there exists a definable set of representatives $X \subseteq A \times G^{2r}$ and a definable bijection $\alpha : \tilde{G}' \to X$. By the definition of the o-minimal topology on $\tilde{G}'$, the map α is continuous on some open subset $U \subseteq \tilde{G}'$ and therefore the composition $j \circ \alpha$, which is just $h_{\tilde{G}}$, is continuous on U as well. Since $h_{\tilde{G}}$ is a group isomorphism it must be continuous everywhere. Because $\tilde{G}'$ is locally compact (and $\tilde{G}$ is Hausdorff) the inverse map is continuous as well. $\square$

In the next section, we investigate each of the three assumptions of Theorem 2.1.

3. Perfect groups

In this section $\mathcal{M}$ can be taken to be an arbitrary o-minimal structure. Recall that G is perfect if $[G, G] = G$.

Claim 3.1. *Let G be a definable group.*
(i) If G is perfect then every homomorphic image of G is perfect.
(ii) The direct product of perfect groups is perfect.
(iii) (Assume that $\langle G, \cdot\rangle$ is sufficiently saturated). If $\tilde{G}$ is definably connected, G perfect and $\pi : \tilde{G} \to G$ is a finite extension then $\tilde{G}$ is perfect as well.
(iv) If G is definably simple then it is perfect.
(v) If G is semisimple and definably connected then it is perfect.

Proof. (i) and (ii) are easy. For (iii), the assumption implies that $\pi[\tilde{G}, \tilde{G}] = G$, and hence $\tilde{G} = F \cdot [\tilde{G}, \tilde{G}]$ for some finite group $F \subseteq \tilde{G}$. It follows that $[\tilde{G}, \tilde{G}]$ is a $\bigvee$-definable group (i.e. a countable union of definable sets) of finite index, hence its complement is also $\bigvee$-definable. This implies, using saturation, that $[\tilde{G}, \tilde{G}]$ is a definable subgroup of finite index, contradicting connectedness.

(iv) If G is definably compact and definably simple then G is elementarily equivalent to a compact simple real Lie group H, by 1.1(1). By topological compactness, there exists an r such that $[H, H]_r = H$. This is now true for G as well.

If G is not definably compact then by 1.1(5) it is abstractly simple and therefore $[G, G] = G$.

(v) Assume that G is semisimple and definably connected. Then $G/Z(G)$ is centerless, definably connected and semisimple. By 1.1(3), the group $G/Z(G)$ can

be written as a direct product of $\mathcal{M}$-definable definably simple groups. the result now follows from (iv), (ii) and (iii). $\square$

4. Property ρ and semisimple groups

In this section we assume that $\mathcal{M}$ is an arbitrary o-minimal structure. Recall that an expansion $\mathbf{G}$ of an $\mathcal{M}$-definable group G is said to have property ρ if the definably connected components of every $\langle G, \cdot\rangle$-definable subset of G^n are definable in $\mathbf{G}$ (possibly over new parameters).

Claim 4.1. *Assume that $G_1, \dots, G_k$ are definable groups, such that the theory of each pure group $\langle G_i, \cdot\rangle$ satisfies property ρ. Then the theory of the pure group $G = G_1 \times \cdots \times G_k$, expanded by a predicate for every $G_i \subseteq G$, satisfies property ρ as well.*

Proof. Every $\langle G, \cdot\rangle$-definable set $X \subseteq G^n$ is a finite union of sets of the form $X_1 \times \cdots \times X_k$, where X_i is a G_i-definable subset of G_i^n. By our assumptions on each G_i, we may assume that each X_i is definably connected. Each definably connected component of X is a finite union of such cartesian products and therefore $\langle G, \cdot\rangle$-definable, together with predicates for every G_i. $\square$

Lemma 4.2. *If G is a definably simple group then the pure group $\langle G, \cdot\rangle$ has property ρ.*

Proof. By 1.1(2), there are two cases to consider: If G is unstable then it is a semialgebraic group which is bi-interpretable (over parameters) with a real closed field. It follows that every $\langle G, \cdot\rangle$-definable set $X \subseteq G^n$ is semialgebraic and every definably connected component of X is again semialgebraic, and therefore $\langle G, \cdot\rangle$-definable (possibly over parameters).

If G is stable then it is a linear algebraic group over a definable algebraically closed field K. Because K is a definable algebraically closed field in the o-minimal structure $\mathcal{M}$, then, by [22], a maximal real closed subfield $R \subseteq K$ is definable in $\mathcal{M}$ and we have $K = R(\sqrt{-1})$. Since G is a linear algebraic group over K, we may assume that $G \subseteq K^\ell$ for some ℓ and that its group-topology agrees with that of K^ℓ, identified with $R^{2\ell}$. In particular, the definably connected components of every definable subset of G^n in the sense of the group topology are the same as those in the sense of the field R.

By 1.1, G is bi-interpretable (again over parameters) with K and hence the $\langle G, \cdot\rangle$-definable subsets of G^n are exactly the K-constructible sets. It is therefore sufficient to prove:

Claim 4.3. *If $K = R(\sqrt{-1})$ is an algebraically closed field definable in an o-minimal $\mathcal{M}$ and $X \subseteq K^n$ is a K-constructible set then every definably connected component of X (in the sense of R) is K-constructible.*

Proof. The set X is of the form

$$X = \bigcup_{i=1}^{r} (X_i \setminus Y_i),$$

with each X_i an irreducible algebraic variety and $Y_i \subseteq X_i$ an algebraic variety of smaller algebraic dimension. We claim that each $X_i \setminus Y_i$ is definably connected.

Indeed, it is known that if $V \subseteq \mathbb{C}^n$ is an irreducible complex variety then $Reg(V)$ (the set of complex regular points of V) is a connected set, dense in V. If we now

work in the structure $\langle \mathbb{R}, <, +, \cdot, \rangle$ then, by quantifying over parameters, this fact carries over to $\langle R, <, +, \cdot, K\rangle$. Thus, every $Reg(X_i)$ is definably connected in the sense of R and dense in X_i.

Thus, the set $Reg(X_i)$ is a definably connected R-manifold of even R-dimension $2k$, and we have $\dim_R(Y_i) \leqslant 2k-2$ (we let $\dim_R(Y_i)$ denote the o-minimal dimension of Y_i with respect to R, which is twice the algebraic dimension of Y_i). The set $Reg(X_i) \setminus Y_i$ is therefore still definably connected, dense in $Reg(X_i)$ and so, also dense in $X_i \setminus Y_i$. It follows that $X_i \setminus Y_i$ is definably connected.

Finally, each definably connected component of X must be a finite union of sets of the form $X_i \setminus Y_i$, so constructible.

With this ends the proof of the claim and of Lemma 4.2 □

Part (ii) of the theorem below follows from a general result by Edmundo, Jones and Peatfield, see [8].

Theorem 4.4. *If $\tilde{G}$ is semisimple and definably connected then*
(i) $\langle \tilde{G}, \cdot\rangle$ is bi-interpretable with $\langle \tilde{G}/Z(\tilde{G}), \cdot\rangle$, after naming a parameter $\bar{b}$ from $\tilde{G}/Z(\tilde{G})$. The parameter can be chosen in $dcl_{\mathcal{N}}(\emptyset)$, for $\mathcal{N} = \langle M, <, \langle G, \cdot\rangle\rangle$ and also in $dcl_{\tilde{G}}(\emptyset)$, where $\tilde{G} = \langle \tilde{G}, \cdot\rangle$.
(ii) There is an $\mathcal{M}$-definable real closed field R, such that the group $\tilde{G}$ is definably isomorphic in $\mathcal{M}$, over parameters, to a semialgebraic group $\tilde{G}'$ over the field of real algebraic numbers $R_{alg} \subseteq R$.
(iii) $\tilde{G}$ has property ρ.

Proof. Consider the extension

$$Z(\tilde{G}) \xrightarrow{\imath} \tilde{G} \xrightarrow{\pi} G.$$

Case I G is definably simple.

(i) By 4.2, the pure group G has property ρ and by 3.1, it is a perfect group. Clearly $\tilde{G}$ has finite center and hence has property Z. Therefore, by Corollary 2.5, the structure $\langle Z(\tilde{G}), \tilde{G}, G, i, \pi\rangle$ is bi-interpretable with $\langle\langle G, \cdot\rangle, \langle Z(\tilde{G}), \cdot\rangle\rangle$, after naming finitely many elements in G and in $Z(\tilde{G})$. Moreover, the finite group $Z(\tilde{G})$ itself can be defined in G after naming finitely many elements (which we may even assume to belong to some 0-definable finite subgroup of G). Thus, $\langle Z(\tilde{G}), \tilde{G}, G, i, \pi\rangle$ is bi-interpretable, over parameters in G, with $\langle G, \cdot\rangle$. Finally, we note that $\langle Z(\tilde{G}), \tilde{G}, G, i, \pi\rangle$ is bi-interpretable with $\langle \tilde{G}, \cdot\rangle$, hence $\langle \tilde{G}, \cdot\rangle$ is bi-interpretable over parameters, with $\langle G, \cdot\rangle$. As was observed in Remark 2.6, the parameters which we use can be taken in $dcl_{\mathcal{N}}(\emptyset)$, where $\mathcal{N} = \langle M, <, \langle G, \cdot\rangle\rangle$.

(ii) By 1.1(1), we may assume that G is an R-semialgebraic group, defined over $R_{alg} \subseteq R$, for some $\mathcal{M}$-definable real closed field R. As was shown above, $\langle \tilde{G}, \cdot\rangle$ is interpretable, now over the empty set, in $\langle R, <, \langle G, \cdot\rangle\rangle$. In particular, the group $\tilde{G}$ is definably isomorphic in $\mathcal{M}$ to a group H which is interpretable in the field R, over R_{alg}. By elimination of imaginaries in real closed fields, H can be chosen to be definable.

(iii) We now want to show that $\tilde{G}$ has property ρ. If G is stable then, by 1.1(2), it is bi-interpretable, over parameters, with an algebraically closed field K. In this case,

because $\tilde{G}$ and G are bi-interpretable, $\tilde{G}$ is definably isomorphic, over parameters, to an algebraic group H over K. By Claim 4.3, if $X \subseteq H^n$ is constructible over K then its definably connected components are constructible over K as well. Because of the bi-interpretability of G (so also of H) with K, these components are definable in $\langle H, \cdot \rangle$, so H (hence $\tilde{G}$) has property ρ.

If G is unstable then it is bi-interpretable with a real closed field R and therefore, by (i), $\tilde{G}$ is also bi-interpretable with R. By (ii), we may assume that $\tilde{G}$ is semialgebraic over R_{alg}. This implies that every semialgebraic subset of $\tilde{G}^n$ is definable in the pure group $\langle \tilde{G}, \cdot \rangle$. In particular, $\tilde{G}$ has property ρ.

Case II G is semisimple.

(i) We first claim that $\tilde{G}$ is bi-interpretable with $G = \tilde{G}/Z(\tilde{G})$ together, possibly with finitely many constants. For that, we need to establish the three assumptions of Theorem 2.1 (with $A = Z(\tilde{G})$):

By 1.1, G is definably isomorphic in $\mathcal{M}$ to a product $H_1 \times \cdots \times H_k$, of definably simple groups. Each of the H_i's is the centralizer of the other $k-1$ groups hence it is definable in the pure group langauge of G (after naming parameters). It follows from 4.2 and 4.1 that G has property ρ. By 3.1, $\tilde{G}$ is perfect. Because $Z(\tilde{G})$ is finite, we clearly have property Z. We can now apply Theorem 2.1 exactly as in Case I.

(ii) This is identical to the proof in Case I.

(iii) We claim that $\tilde{G}$ has property ρ:

Assume that $G = H_1 \times \cdots \times H_k$, for definably simple H_i's and let $\tilde{H}_i$ be the pull back of H_i under the inverse image of $\pi : \tilde{G} \to G$. Each $\tilde{H}_i$ is a finite central extension of H_i, and if we let $\tilde{H} = \tilde{H}_1 \times \cdots \times \tilde{H}_k$ and $\tilde{\pi} : \tilde{H} \to G$ be the natural projection, then $\tilde{\pi}$ factors through the finite extensions $\pi' : \tilde{H} \to \tilde{G}$ and $\pi : \tilde{G} \to G$.

$$\begin{tikzcd} \tilde{H} \arrow[rr, "\pi'"] \arrow[dr, "\tilde{\pi}"'] & & \tilde{G} \arrow[dl, "\pi"] \\ & G & \end{tikzcd}$$

Claim $\tilde{H}$ has property ρ.

Proof. Each $\tilde{H}_i$ is a finite central extension of a definably simple group so by Case I, it has property ρ. Therefore, by 4.1, in order to see that $\tilde{H}$ itself has property ρ it is sufficient to see that each $\tilde{H}_i$, $i = 1, \dots, k$, is definable in the pure group $\langle \tilde{H}, \cdot \rangle$, possibly with parameters. Let us see why $\tilde{H}_1$ is definable. First note that the centralizer of $\tilde{H}_2 \cup \cdots \cup \tilde{H}_k$, call it Z_1, is $\tilde{H}_1 \cdot Z(\tilde{H})$ (where $Z(\tilde{H})$ is finite). The group Z_1 is a definable, possibly disconnected, group and it is sufficient to see that we can define in the pure group structure, its connected component. By 3.1(v), $\tilde{H}_1$ is perfect, and it is easy to see that $[Z_1, Z_1] \subseteq \tilde{H}_1$. Hence, $\tilde{H}_1 = [Z_1, Z_1]_k$ for some k and this last group is clearly definable. End of Claim.

Because G is perfect and has property ρ, we can apply Theorem 2.1 to the finite central extension $\tilde{\pi} : \tilde{H} \to G$ and conclude that the pure groups G and $\tilde{H}$ are bi-interpretable, after naming constants from G.

Let $X \subseteq \tilde{G}^n$ be a $\tilde{G}$-definable set and let $X_1, \dots, X_k$ be its definably connected components, with respect to the $\tilde{G}$-topology. Because $\tilde{G}$ and $\pi' : \tilde{H} \to \tilde{G}$ are definable in $\langle \tilde{H}, \cdot \rangle$ and continuous, the set $Y = \pi'^{-1}(X)$ is definable in $\tilde{H}$ and each

$\pi'^{-1}(X_i)$ is a finite union of definably connected components of Y, hence definable in $\tilde{H}$ (because $\tilde{H}$ has property ρ). It follows that each X_i is definable in $\tilde{H}$. However, as we already saw, $\tilde{H}$ and G are bi-interpretable and G and $\tilde{G}$ are bi-interpretable as well, and therefore each X_i is definable in $\tilde{G}$ (after possibly naming finite many parameters). Hence, $\tilde{G}$ has property ρ. □

5. Property Z

Assume now that $\mathcal{M}$ expands a real closed field R, in a neighborhood of the identity of a definable group G. We denote by $\mathcal{G}$ its Lie algebra whose underlying R-vector space is the tangent space of G at e, $T_e(G)$. We recall some facts about groups and Lie algebras, as presented in [18].

Assume that G is definably connected. To every definable subgroup $H \subseteq G$ there is an associated Lie subalgebra $\mathfrak{h} \subseteq \mathcal{G}$. The subgroup H is normal in G if and only if $\mathfrak{h}$ is an ideal in $\mathcal{G}$ (see [18, Theorem 2.32]). For every $g \in G$, we denote by $Ad_g : T_e(G) \to T_e(G)$ the differential of the inner automorphism $a_g : x \mapsto gxg^{-1}$. If $\mathcal{G}_1$ is a linear subspace of $\mathcal{G}$ then $\mathcal{G}_1$ is an ideal if and only if it is invariant under Ad_g for all $g \in G$. (See Claim 2.31 there).

If H is a locally definable subgroup of G (e.g. when $H = [G, G]$) then, just as for definable subgroups, one can associate to H a Lie subalgebra $L(H) \subseteq \mathcal{G}$. If H is normal in G then $L(H)$ is an ideal in $\mathcal{G}$ (indeed, because the whole analysis is local in nature, it is enough to consider H at a neighborhood of e and in this case the arguments work as in the definable case).

Recall that a subalgebra $\mathcal{A} \subseteq \mathcal{G}$ is called central if for every every $\xi \in \mathcal{A}$ and $\eta \in \mathcal{G}$, we have $[\xi, \eta] = 0$. Equivalently (see [18, Corollary 2.32]), for every $g \in G$, $Ad_g|\mathcal{A} = id$.

Theorem 5.1. *Let G be a definably connected group. Assume that $\mathcal{G} = \mathcal{A} + \mathcal{I}$ for a central subalgebra $\mathcal{A}$, and an ideal $\mathcal{I} \subseteq \mathcal{G}$. Then $L([G, G]) \subseteq \mathcal{I}$.*

Proof. We first introduce some notation. Let $\mathfrak{h} = L([G, G])$ and for $g \in G$, let $\ell_g : G \to G$ be left-multiplication by g and $r_g : G \to G$ be right-multiplication by g. For every $g \in G$ we have

$$T_g(G) = d_e(\ell_g)(T_e(G)) = d_e(r_g)(T_e(G)),$$

and similarly, for every $h \in [G, G]$ we have

$$T_h([G, G]) = d_e(\ell_g)(\mathfrak{h}) = d_e(r_g)(\mathfrak{h}).$$

It is therefore sufficient to prove, for some $h \in [G, G]$, that $d_h(r_h^{-1})(T_h([G, G])) \subseteq \mathcal{I}$.

Because $[G, G]$ is locally-definable there exists an n and an open neighborhood $U \subseteq G$ of e such that $U \cap [G, G] = U \cap [G, G]_n$. Consider the function $F_n = F_{n,G} : G^{2n} \to G$ as given earlier, by the product of n-many group commutators. It is not hard to see that for sufficiently generic $\bar{g} = (g_1, \dots, g_{2n})$ in $F_n^{-1}(U \cap [G, G])$, we have

$$d_{\bar{g}}(F_n)(T_{g_1}(G) \times \cdots \times T_{g_{2n}}(G)) = T_{F_n(\bar{g})}([G, G]).$$

Using the chain rule, it is sufficient to prove that for every $\bar{g} \in G^{2n}$, we have

$$d_{\bar{g}}(r^{-1}_{F_n(\bar{g})} \circ F_n)(T_{g_1}(G) \times \cdots \times T_{g_{2n}}(G)) \subseteq \mathcal{I}.$$

We are going to prove this by induction on n. For that purpose, let us a call definable function $\alpha : G^k \to G$ *good at* $\bar{g} \in G^k$ if it satisfies

$$d_{\bar{g}}(r_{\alpha(\bar{g})^{-1}} \circ \alpha)(T_{g_1}(G) \times \cdots \times T_{g_k}(G)) \subseteq \mathcal{I}.$$

Claim (i) If $\alpha : G^k \to G$ is good at $\bar{g}$ and $\alpha(\bar{g}) = e$ then for every $h \in G$, the function $a_g \circ \alpha$ is good (recall $a_g(x) = gxg^{-1}$).

(ii) If $\alpha, \beta : G^k \to G$ are good at $\bar{g} \in G^k$ then so is $\alpha \cdot \beta$, the group product of the two.

Proof (i) This is immediate from the fact that $\mathcal{I}$ is invariant under $d_e(a_g) = Ad_g$.

(ii) If $\mu : G \times G \to G$ is the group product then $d_{(e,e)}(\mu) = (id, id)$. Now, in the special case that $\alpha(\bar{g}) = \beta(\bar{g}) = e$ we have, by the chain rule, $d_{\bar{g}}(\mu(\alpha, \beta)) = d_{\bar{g}}(\alpha) + d_{\bar{g}}(\beta)$, and therefore $\mu(\alpha, \beta)$ is good at $\bar{g}$.

More generally, if $\alpha(\bar{g}) = h_1, \beta(\bar{g}) = h_2$ then

$$\alpha(\bar{x})\beta(\bar{x})h_2^{-1}h_1^{-1} = \alpha(\bar{x})h_1^{-1}(h_1(\beta(\bar{x})h_2^{-1})h_1^{-1}),$$

and hence

$$r^{-1}_{(h_1h_2)} \circ \mu(\alpha, \beta) = \mu(r_{h_1^{-1}} \circ \alpha, a_{h_1} \circ r_{h_2^{-1}} \circ \beta).$$

By definition, $r_{h_1^{-1}} \circ \alpha$ and $r_{h_2^{-1}} \circ \beta$ are good at $\bar{g}$ and the two functions send $\bar{g}$ to e. By (i) and the special case we just did, $\mu(\alpha, \beta)$ is good at $\bar{g}$ as well. End of Claim.

Because every F_n is a product of commutators, it is sufficient, using Claim (ii) above, to prove that $F_1(x, y) = xyx^{-1}y^{-1}$ is good at every $(g, h) \in G^2$. Because $F_1(g, h) = ghg^{-1}h^{-1}$, we need to show that $r_{hgh^{-1}g^{-1}} \circ F_1$ is good at (g, h), or equivalently, that $\sigma(x, y) = r_{hgh^{-1}g^{-1}} \circ F_1(gx, hy)$ is good at (e, e). Rewriting $\sigma(x, y)$ we get:

$$\begin{aligned} gxhyx^{-1}g^{-1}y^{-1}h^{-1}hgh^{-1}g^{-1} \quad &= gxg^{-1}gh(yx^{-1}g^{-1}y^{-1}g)(gh)^{-1} \\ &= a_g(x) \cdot a_{gh}(y \cdot x^{-1} \cdot a_{g^{-1}}(y^{-1})). \end{aligned}$$

The right-most expression can be re-written as

$$a_g(x) \cdot a_{gh}(y) \cdot a_{gh}(x)^{-1} \cdot a^{-1}_{ghg}(y)^{-1}.$$

We have a product of four functions, each sending e to e. Taking the differential and applying the chain rule we obtain, for every $u, v \in T_e(G)$:

$$d_{(e,e)}\sigma(u, v) = Ad_g(u) + Ad_{gh}(v) - Ad_{gh}(u) - Ad_{ghg^{-1}}(v).$$

We now return to our assumptions. Every $u \in \mathcal{G}$ can be written as $u = u_1 + u_2$, where $u_1 \in \mathcal{A}$ and $u_2 \in \mathcal{I}$. Because $\mathcal{A}$ is central we have $Ad_g(u_1) = u_1$ for every $g \in G$. Hence, $d_{(e,e)}\sigma(u_1 + u_2, v_1 + v_2)$ equals to:

$$\begin{aligned} &u_1 + Ad_g(u_2) + v_1 + Ad_{gh}(v_2) - u_1 - Ad_{gh}(u_2) - v_1 - Ad_{ghg^{-1}}(v_2) = \\ &= Ad_g(u_2) + Ad_{gh}(v_2) - Ad_{gh}(u_2) - Ad_{ghg^{-1}}(v_2). \end{aligned}$$

Because $\mathcal{I}$ is invariant under every Ad_g and under $+$, the sum on the right belongs to $\mathcal{I}$. $\square$

Corollary 5.2. *Let G be a definably connected group, $A \subseteq G$ a definable central subgroup and let $\mathcal{A}$ be the Lie algebra of A. Assume that $\mathcal{G}$ can be written as a direct sum $\mathcal{G} = \mathcal{A} \oplus \mathcal{I}$, for some ideal $\mathcal{I}$. Then for every n, $A \cap [G, G]_n$ is finite.*

Proof. The Lie algebra $\mathcal{A}$ is central in $\mathcal{G}$ (see [18, Claim 2.32]). By our assumption, and Theorem 5.1 we have $L([G,G]) \subseteq \mathcal{I}$ and therefore $\mathcal{A} \cap L([G,G]) = \{0\}$. Because $A \cap [G,G]$ is a locally definable group it has a Lie Algebra of the same dimension, which equals $\mathcal{A} \cap L([G,G])$. Hence, $\dim(A \cap [G,G]) = 0$ and therefore $A \cap [G,G]_n$ is finite for every n. □

Corollary 5.3. *Let $\tilde{G}$ be a definably connected central extension of a semisimple group G, with $L(\tilde{G}) = \tilde{\mathcal{G}}$. Then*
(i) For every n, the set $Z(\tilde{G}) \cap [\tilde{G},\tilde{G}]_n$ is finite.
(ii) The Lie algebra of the locally definably group $[\tilde{G},\tilde{G}]$ equals to $[\tilde{\mathcal{G}},\tilde{\mathcal{G}}]$ and we have $\tilde{\mathcal{G}} = \mathcal{Z} \oplus L([\tilde{G},\tilde{G}])$, where $\mathcal{Z} = L(Z(\tilde{G}))$. Moreover,

$$L([\tilde{G},\tilde{G}]) \simeq L(G).$$

Proof. (i) First note that $\mathcal{Z} = L(Z(\tilde{G}))$ is the center of $\tilde{\mathcal{G}}$ (see [18, Claim 2.31]) and the Lie algebra of $\tilde{G}/Z(\tilde{G})$ equals $\tilde{\mathcal{G}}/\mathcal{Z}$. It follows that $\tilde{\mathcal{G}}/\mathcal{Z}$ is a semisimple Lie algebra (see Theorem 2.34 there).

By the Levi decomposition theorem for Lie algebras, $\tilde{\mathcal{G}}$ is the semi-direct product of its solvable radical and a semisimple Lie sub-algebra $\mathfrak{h}$. Because $\tilde{\mathcal{G}}/\mathcal{Z}$ is semisimple it follows that $\mathcal{Z}$ is the solvable radical of $\tilde{\mathcal{G}}$. We claim that $\mathfrak{h} = [\tilde{\mathcal{G}},\tilde{\mathcal{G}}]$.

Indeed, for $\xi_i = \xi_i + \eta_i \in \tilde{\mathcal{G}}$, $i = 1, 2$, and $\xi_i \in \mathcal{Z}$ and $\eta_i \in \mathfrak{h}$, we have

$$[\xi_1, \xi_2] = [\xi_1, \xi_2 + \eta_2] + [\eta_1, \xi_2] + [\eta_1, \eta_2] = [\eta_1, \eta_2] \in \mathfrak{h}.$$

It follows that $[\tilde{\mathcal{G}},\tilde{\mathcal{G}}] \subseteq \mathfrak{h}$ and because $\mathfrak{h}$ is semisimple we also have $\mathfrak{h} = [\mathfrak{h},\mathfrak{h}] \subseteq [\tilde{\mathcal{G}},\tilde{\mathcal{G}}]$, and hence $\mathfrak{h} = [\tilde{\mathcal{G}},\tilde{\mathcal{G}}]$. Therefore, $\mathfrak{h}$ is an ideal in $\tilde{\mathcal{G}}$ and we have $\tilde{\mathcal{G}} = \mathcal{Z} \oplus [\tilde{\mathcal{G}},\tilde{\mathcal{G}}]$.

We can now apply Corollary 5.2 and conclude that $Z(\tilde{G}) \cap [\tilde{G},\tilde{G}]_n$ is finite for every n.

(ii) By dimension considerations, the above implies that $\dim L([\tilde{\mathcal{G}},\tilde{\mathcal{G}}]) = \dim([\tilde{G},\tilde{G}]) = \dim(\tilde{\mathcal{G}}) - \dim Z(\tilde{G}) = \dim[\tilde{\mathcal{G}},\tilde{\mathcal{G}}]$. By Theorem 5.1, $L([\mathcal{G},\mathcal{G}]) \subseteq [\tilde{\mathcal{G}},\tilde{\mathcal{G}}]$, and therefore $L([\mathcal{G},\mathcal{G}]) = [\tilde{\mathcal{G}},\tilde{\mathcal{G}}]$.

Again, by dimension considerations, $\dim([\tilde{G},\tilde{G}]) = \dim G$ and hence $d\pi$ is an isomorphism of $L([\tilde{G},\tilde{G}])$ and $L(G)$. □

Because every definably compact definably connected group is a central extension of a semisimple group (1.1) we immediately conclude the result below. As we will later see (Corollary 6.4), this is only a first approximation to the stronger theorem about the commutator subgroup of a definably compact group.

Corollary 5.4. *Let $\tilde{G}$ be a definably connected definably compact group. Then for every n, $Z(\tilde{G}) \cap [\tilde{G},\tilde{G}]_n$ is finite.*

5.1. **Omitting the real closed field assumption.** The real closed field assumption was of course necessary for the discussion in the last section, because it involved the Lie algebra of $\tilde{G}$ at e. However, once used this assumption can be weakened, at least in the definably compact case.

We first recall some notions: An o-minimal expansion of an ordered group is called *semi-bounded* if there is no definable bijection between bounded and unbounded intervals. There are three different possibilities for an o-minimal expansion $\mathcal{M}$ of an ordered group (see discussion in [17]):
1. $Th(\mathcal{M})$ is linear, i.e. $\mathcal{M}$ is elementarily equivalent to an ordered reduct of an ordered vector space over an ordered division ring.

2. $Th(\mathcal{M})$ is not linear and not semi-bounded, in which case there exists a definable real closed field whose domain is M.
3. $\mathcal{M}$ is semi-bounded and $Th(\mathcal{M})$ is not linear.

We can now state the following generalization of Corollary 5.3:

Corollary 5.5. *Let $\mathcal{M}$ be an o-minimal expansion of an ordered group and let G be an $\mathcal{M}$-definable, definably compact, definably connected group. Then for every $n \in \mathbb{N}$, $Z(G) \cap [G,G]_n$ is finite.*

Proof. We need to examine the three cases above. If $\mathcal{M}$ is elementarily equivalent to a reduct of an ordered vector space then every definable group is abelian-by-finite (indeed, by [21], if not then a field is definable in $\mathcal{M}$. It is easy to see that this is impossible), so G is abelian.

If $\mathcal{M}$ is not linear and not semi-bounded then $\mathcal{M}$ expands a real closed field and therefore Corollary 5.3 applies. We are thus left with the semi-bounded nonlinear case. Recall the following from [17]:

If $\mathcal{M}$ is semi-bounded and $Th(\mathcal{M})$ is not linear then there exists $\mathcal{N} \succ \mathcal{M}$ and an o-minimal expansion $\widehat{\mathcal{N}}$ of $\mathcal{N}$ (by "expansion" we mean here that every definable subset of $\mathcal{N}$ is definable in $\widehat{\mathcal{N}}$, possibly with additional parameters) and an elementary substructure $\widehat{D} \prec \widehat{\mathcal{N}}$ such that every interval in $\widehat{D}$ admits a definable real closed field. Given G, an $\mathcal{M}$-definable, definably connected and definably compact group, we can view G as an $\mathcal{N}$-definable (and therefore also $\widehat{\mathcal{N}}$-definable) group. Hence, there exists in $\widehat{\mathcal{N}}$ a 0-definable family $\mathcal{F} = \{G_s : s \in S\}$ of groups, all definably compact and definably connected such that $G = G_{s_0}$ for some $s_0 \in S$. Furthermore, the domain of every such G_s is a bounded subset of $\widehat{\mathcal{N}}$ (see [17]). Given a fixed $n \in \mathbb{N}$, we can now argue in $\widehat{D}$:

For every $s \in S(\widehat{D})$, because G_s is bounded, its underlying set is contained in R^n, for some definable real closed field R in $\widehat{D}$. The field R, with all its $\widehat{D}$-induced structure is o-minimal, and therefore, Corollary 5.3 applies and hence $Z(G_s) \cap [G_s, G_s]_n$ is finite. Since this is true for every $s \in S(\widehat{D})$, there exists a bound $k = k(n)$ such that $\widehat{D} \models \forall s \in S \ |Z(G_s) \cap [G_s, G_s]_n| \leqslant k$. This is a first order statement which carries over to $\widehat{\mathcal{N}}$ and therefore to $\mathcal{N}$ as well. It follows that $Z(G) \cap [G,G]_n$ is finite. $\Box$

Question. Is there a direct proof, avoiding the Lie algebra argument for the following result: If $\tilde{G}$ is a definably connected central extension of a semisimple group in an arbitrary o-minimal structure then the set $Z(\tilde{G}) \cap [\tilde{G}, \tilde{G}]_n$ is finite for every n?

6. The main results

6.1. **Interpretability results.**

Theorem 6.1. *Let $\mathcal{M}$ be an o-minimal expansion of a real closed field and let $A \to \tilde{G} \to G$ be a definable central extension of a semisimple group G, with $\tilde{G}$ definably connected. Then*

(1) *$\langle A, \tilde{G}, i, \pi \rangle$ is bi-interpretable, over an imaginary parameter $\bar{c}$, with $\langle G, A \rangle$. The parameter $\bar{c}$ names a map from a family of definably connected components of a 0-definable set in $\langle G, \cdot \rangle$ onto a finite subset of A.*

(2) *The exact sequence* $A \to \tilde{G} \to G$ *is elementarily equivalent, after naming parameters on both sides, to a semialgebraic central extension* $A' \to \tilde{G}' \to G'$*, defined over the real algebraic numbers, with* $\dim(\tilde{G}') = \dim(\tilde{G})$.

If $\tilde{G}$ *is definably compact then, in both (1) and (2), it is sufficient to assume that* $\mathcal{M}$ *expands an ordered group. In this case* $\tilde{G}'$ *of (2) can be chosen definably compact as well.*

An immediate corollary of (2) above is:

Corollary 6.2. *If* $\mathcal{M}$ *is an o-minimal expansion of an ordered group then every definably compact, definably connected group is elementarily equivalent, in pure group language, to a compact semialgebraic (in particular Real Lie) group over the real algebraic numbers.*

Proof of Theorem 6.1:
(1) We need only to establish the three assumptions of Theorem 2.1.
By Theorem 4.4, $\langle G, \cdot\rangle$ has property ρ with respect to the pure group structure. By Corollary 5.3 (and Corollary 5.5 in the definably compact case), $\tilde{G}$ has the property Z. By Claim 3.1, G is perfect. □

(2) By (1), $\langle A, \tilde{G}, G, i, \pi\rangle$ is bi-interpretable with $\langle G, A\rangle$, after naming the necessary map. By 4.4, G itself is definably isomorphic in $\mathcal{M}$ to a semialgebraic group G' over the real algebraic numbers, which is clearly definably compact if G is. Therefore, in order to prove (2), it is sufficient to show: Given a finite set $C \subseteq A$, the structure $\langle A, +, C\rangle$ is elementarily equivalent to $\langle A', +, C'\rangle$ for some semialgebraic group A' over the real algebraic numbers, and a finite subset $C' \subseteq A'$. This is exactly the content of Claim 11.3 in the Appendix. □

Remark As the proof of (2) above shows, the only obstacle for a definable central definably connected extension $\tilde{G}$ of a definable semisimple group to be definably *isomorphic* to a semialgebraic group is the group $Z(\tilde{G})$. Hence, if $Z(\tilde{G})$ is definably isomorphic in $\mathcal{M}$ to a semialgebraic group then so is G. We can also prove an analogue for algebraic groups:

Corollary 6.3. *Assume that* $\mathcal{M}$ *is an o-minimal expansion of a real closed field* R*. Let* $A \to \tilde{G} \to G$ *be a definable central extension of a definably connected semisimple group* G*.*

If G *is a stable group and* A *is definably isomorphic in* $\mathcal{M}$ *to an algebraic group over* $K = R(\sqrt{-1})$*, then* $\tilde{G}$ *is definably isomorphic in* $\mathcal{M}$ *to an algebraic group over* K*.*

Proof. Since G is stable, $G/Z(G)$ is a direct product of definably simple stable groups, which we may assume are all algebraic groups over K (see 1.1). Hence, $G/Z(G)$ is definably isomorphic to an algebraic group over K. By Theorem 4.4(1), G is definable, possibly over parameters, in the group $G/Z(G)$ and therefore it is definably isomorphic in $\mathcal{M}$ to algebraic group over K. We continue as in the proof of Theorem 6.1. □

We end this discussion with an example showing that not every definably connected group in an o-minimal structure is elementarily equivalent to a real Lie group which is definable in an o-minimal structure. This is a small variation of an example

from [20], so we will be brief:

Example Let $\mathcal{M} = \langle R, <, +, \cdot, exp\rangle$ be a nonstandard model of theory of the real exponential field, and let $\alpha \in R$ be element greater than all natural numbers. We define

$$G = \left\{ \begin{pmatrix} t & 0 & u \\ 0 & t^\alpha & v \\ 0 & 0 & 1 \end{pmatrix} : u, v \in M, t > 0 \right\}.$$

The group G is a solvable centerless group, and as is shown in [20, p.4], the structure $\mathcal{M}_\alpha = \langle R, <, +, \cdot, t \mapsto t^\alpha\rangle$ is interpretable in the pure group G. If G were elementarily equivalent to a definable real Lie group H in some o-minimal structure over the reals then H would interpret a structure $\mathcal{N}_\alpha \equiv \mathcal{M}_\alpha$ so the underlying field of $\mathcal{N}_\alpha$ is non-archimedean. However, every real closed field which is interpretable in an o-minimal structure over the reals must be archimedean (its ordering is Dedekind complete). Contradiction.

6.2. **Structural results.** We can now deduce a structural result about definably compact groups in o-minimal structures.

Corollary 6.4. *Let $\mathcal{M}$ be an o-minimal expansion of an ordered group. Let $\tilde{G}$ be a definably compact, definably connected group. Then*
(i) $[\tilde{G}, \tilde{G}]$ is definable, definably connected and semisimple.
(ii) $\tilde{G}$ equals the almost direct product of $[\tilde{G}, \tilde{G}]$ and $Z(\tilde{G})^0$. Namely, $\tilde{G}$ is the group product of $[\tilde{G}, \tilde{G}]$ and $Z(\tilde{G})^0$ and the intersection of two groups is finite. In particular, $\tilde{G} \simeq (Z(\tilde{G})^0 \times [\tilde{G}, \tilde{G}])/F$, for a finite central subgroup $F \subseteq Z(\tilde{G})^0 \times [\tilde{G}, \tilde{G}]$.

Proof. (i) By 6.2, $\tilde{G}$ is elementarily equivalent to a compact real Lie group H. By classical Lie group theory, $[H, H]$ is a closed connected semisimple subgroup of H (Indeed, this can be found, for example, in [16, Chapter 5.2, Theorem 4]).

By topological compactness, there exists a k such that $[H, H]_k = H$, i.e. the set $[H, H]_k$ is already a subgroup of H. It follows that the same is true for $\tilde{G}$ hence $[\tilde{G}, \tilde{G}] = [\tilde{G}, \tilde{G}]_k$ is definable. The group $[\tilde{G}, \tilde{G}]$ is definably connected as the continuous image of the definably connected group $\tilde{G}$.

(ii) Because the intersection of $Z(\tilde{G})$ and $[\tilde{G}, \tilde{G}]$ is 0-dimensional (Corollary 5.3), it must be finite. the final clause is immediate from the fact that $[\tilde{G}, \tilde{G}]$ acts trivially on $Z(\tilde{G})^0$, and hence the map $(g, h) \mapsto gh$ from $Z(\tilde{G})^0 \times [\tilde{G}, \tilde{G}]$ to $\tilde{G}$ is a homomorphism with finite kernel. □

Remark Actually, by a theorem of Goto, [12, Theorem 6.55], every element of $[H, H]$ is a commutator (i,e, $[H, H] = [H, H]_1$) hence the same is true in every definably compact group.

Corollary 6.5. *Let $\mathcal{M}$ be an o-minimal expansion of an ordered group and assume that $\tilde{G}$ is definably connected with $\tilde{G}/Z(\tilde{G})$ semisimple and definably compact. Then*
(i) $[\tilde{G}, \tilde{G}]$ is definable.
(ii) $\tilde{G} \simeq (Z(\tilde{G})^0 \times [\tilde{G}, \tilde{G}])/F$, for a finite central subgroup $F \subseteq Z(\tilde{G})^0 \times [\tilde{G}, \tilde{G}]$

Proof. By 6.1 (2), the assumption implies that $Z(G)^0 \to \tilde{G} \to G$ is elementarily equivalent to a semialgebraic extension $Z_0 \to \tilde{G}_0 \to G_0$ over the real numbers, with G_0 a compact connected semisimple real Lie group. By Corollary 5.3 (ii), we have

$L(\tilde{G}_0) \simeq L(Z_0) \oplus [L(\tilde{G}_0), L(\tilde{G}_0)]$, and the Lie algebra of the locally definable group $[\tilde{G}_0, \tilde{G}_0]$ equals $[L(\tilde{G}_0), L(\tilde{G}_0)]$. Moreover, $[L(\tilde{G}_0), L(\tilde{G}_0)]$ is isomorphic to $L(G_0)$.

We first claim that $[\tilde{G}_0, \tilde{G}_0]$ is a compact subgroup of $\tilde{G}_0$.

We recall the following definition: A Lie algebra over $\mathbb{R}$ is called *compact* if it admits an invariant positive definite scalar product. Clearly, a subalgebra of a compact Lie algebra is also compact and if a Lie algebra is commutative then it is compact (any positive definite scalar product will do). Furthermore, the direct sum of two compact Lie algebras is compact as well. Finally, the Lie algebra of a compact real Lie group is a compact, see [16, p. 228, 12].

Because G_0 is compact, its Lie algebra is compact. Hence $L(G_0) = [L(\tilde{G}_0), L(\tilde{G}_0)]$ is a compact Lie algebra as well. Because $L(Z_0)$ is abelian it is also compact. It follows that $L(\tilde{G}_0)$ is a compact Lie algebra as well. We now apply a theorem about connected Lie groups with compact Lie algebras (see [16, p. 242, Theorem 5]) and conclude that, as a Lie group, $\tilde{G}_0 = B \times C[\tilde{G}_0, \tilde{G}_0]$, for Lie subgroups $B, C \subseteq Z(\tilde{G})^0$ (B torsion-free and C compact), and with $C \cdot [\tilde{G}_0, \tilde{G}_0]$ a compact Lie subgroup, which we denote by H (these groups are not claimed to be definable).

Because $L([\tilde{G}_0, \tilde{G}_0])$ is a maximal semisimple subalgebra of $L(H)$ it follows, using Levi decomposition theorem, that $L(H) = L(C) \oplus L([\tilde{G}_0, \tilde{G}_0])$. As we saw before, since H is compact, the group $[H, H]$ is a compact subgroup of H, which in this case must equal $[\tilde{G}_0, \tilde{G}_0]$. Thus, $[\tilde{G}_0, \tilde{G}_0]$ is a compact subgroup of $\tilde{G}_0$.

It follows that there is a k (actually, as remarked above, $k = 1$), such that $[\tilde{G}_0, \tilde{G}_0]_k = [\tilde{G}_0, \tilde{G}_0]$. By elementary equivalence, $[\tilde{G}, \tilde{G}]_k = [\tilde{G}, \tilde{G}]$, hence this group is definable.

As we already saw, see 5.3, the intersection of $[\tilde{G}, \tilde{G}]$ and $Z(\tilde{G})$ has zero dimension and therefore is finite. Hence, $\tilde{G} \simeq (Z(\tilde{G}^0) \times [\tilde{G}, \tilde{G}])/F$ for a finite central subgroup F. □

7. The connection to Pillay's conjecture

Corollary 6.2, gives a strong connection between definably compact, definably connected groups in o-minimal structures and compact real Lie groups.

Pillay's Conjecture (now a theorem, for expansions of ordered groups, see [13] and [14]) suggests another such connection to compact real Lie groups:

Let G be a definably compact group in a sufficiently saturated o-minimal structure. There exists a minimal type-definable subgroup $G^{00} \subseteq G$ such that G/G^{00}, equipped with the logic topology, is isomorphic to a real Lie group, and the topological dimension of G/G^{00} equals the o-minimal dimension of G.

Our goal is to prove, in the definably connected case, that the pure groups G and G/G^{00} are elementarily equivalent. More precisely, we will prove:

Theorem 7.1. *Let $\mathcal{M}$ be a sufficiently saturated o-minimal expansion of an ordered group and let G be a definably compact, definably connected group. Then*

$$\langle G, \cdot \rangle \equiv \langle G/G^{00}, \cdot \rangle.$$

Moreover, the map $\pi : G \to G/G^{00}$ "splits elementarily", namely there exists an elementary embedding (with respect to the group structure) $\sigma : G/G^{00} \to G$ which is also a section for π.

By Corollary 6.4, definably compact groups can be analyzed using abelian and semisimple subgroups. We first handle the abelian case. Because we are going later on to treat definable groups which are not necessarily connected, we will need to work in a more general setting of abelian groups together with finitely many automorphisms.

7.1. **Definable abelian groups with an additional abelian structure.** Let A be an abelian group definable in an o-minimal structure $\mathcal{M}$, which we assume to expand an ordered group. We denote by $\mathbf{A}_{ab}$ the expansion of the group A by all $\mathcal{M}$-definable subgroups of A^n, $n \in \mathbb{N}$, and let L_{ab} be the associated language (note that if A is definably compact and abelian then, by [21, Cor. 5.2], every $\mathcal{M}$-definable subgroup of A is actually 0-definable in $\mathcal{M}$). For B a subgroup of A we let $\mathbf{B}_{ab}$ be the L_{ab}-substructure of $\mathbf{A}_{ab}$.

In the appendix we treat the general (not necessarily o-minimal) such situation and observe, using known results:

Fact 7.2. *Let A be a abelian definable group in an o-minimal structure, then*

(1) *The structure $\mathbf{A}_{ab}$ eliminates quantifiers.*
(2) *Assume that $B \leqslant A$ is an arbitrary subgroup of A.*
Then $\mathbf{B}_{ab} \prec \mathbf{A}_{ab}$ if and only if the following hold:
(i) For every 0-definable (in $\mathbf{A}_{ab}$) subgroup $S \leqslant A^{n+k}$ and $b \in B^k$,
$$S(B^n, b) \neq \emptyset \Leftrightarrow S(A^n, b) \neq \emptyset$$
and (ii) For every 0-definable (in $\mathbf{A}_{ab}$) subgroups $S_1 \leqslant S_2 \leqslant A^n$,
$$[S_2 : S_1] = [S_2 \cap B^n : S_1 \cap B^n],$$
with the meaning that if this index is infinite on one side then it is infinite on the other.
(3) *If $\mathbf{B}_{ab} \prec \mathbf{A}_{ab}$, for a subgroup B of A, then there exists a surjective group homomorphism $\phi : A \to B$ which is the identity map on B and in addition sends every 0-definable group $S \subseteq A^n$ onto $S \cap B^n$. (We call such a map ϕ* a homomorphic retract*).*

Lemma 7.3. *Let A be a definable compact, definably connected abelian group, in an o-minimal expansion $\mathcal{M}$ of an ordered group and let $\mathbf{A}_{ab}$ be as above.*

(1) *Assume that $\mathbf{B}_{ab} \prec \mathbf{A}_{ab}$ for a subgroup B of A, $\phi : A \to B$ a homomorphic retract and let $A_1 = ker\phi$. Then $(\mathbf{Tor(A)} + \mathbf{A_1})_{ab} \prec \mathbf{A}_{ab}$.*
(2) *If $\mathbf{A}_{ab}$ is sufficiently saturated then $(\mathbf{Tor(A)} + \mathbf{A^{00}})_{ab} \prec \mathbf{A}_{ab}$.*

Note that if we take, in (1) above, $B = A$ and ϕ the identity map then the lemma implies in particular that $\mathbf{Tor(A)}_{ab} \prec \mathbf{A}_{ab}$.

Proof. (1) We need to see that $Tor(A) + A_1$ satisfies the two requirements of 7.2(2). Note that B is divisible and contains all torsion elements of A (there are finitely many for each exponent). Therefore, since $A = B \oplus A_1$, the group A_1 is divisible as well (see 11.1).

Clause (i): Let S be an $\mathcal{M}$-definable subgroup of A^{n+k} and assume that $(b + a_1, c) \in S$ for some $b \in B^n, a_1 \in A_1^n$ and $c \in (Tor(A) + A_1)^k$. We want to show that there is $a \in (Tor(A) + A_1)^n$ such that $(a, c) \in S$.

We write $c = c_1 + c_2$, for $c_1 \in Tor(A)^k$ and $c_2 \in A_1^k$. If $mc_1 = 0$ then
$$m(b + a_1, c) = m(b + a_1, c_1 + c_2) = (mb + ma_1, mc_2) \in S,$$

with $(ma_1, mc_2) \in A_1^{n+k}$. Because ϕ is a retract, we have $(mb, 0) \in S$. Now, if k is the index of $S(A^n, 0)^0$ in $S(A^n, 0)$ then $(kmb, 0) \in S(A^n, 0)^0$. Because this last group is divisible, there exists $(a, 0) \in S(A^n, 0)^0$ such that $kma = kmb$ and therefore $b - a \in Tor(A)^n$. Finally, we have $(b + a_1, c) - (a, 0) = (b - a + a_1, c) \in S$, with $b - a + a_1 \in (Tor(A) + A_1)^n$, as needed.

Clause (ii): We will actually prove a stronger statement than needed here:

(*) If $C \subseteq A^n$ is a divisible subgroup containing $Tor(A)^n$ then for every $S_1 \subseteq S_2 \subseteq A^n$ definable groups.

$$[S_2 : S_1] = [S_2 \cap C : S_1 \cap C]$$

(and both are infinite if one of them is).

We clearly have $[S_2 \cap C : S_1 \cap C] \leqslant [S_2 : S_1]$, so we only need the opposite inequality.

Assume first that S_1 is definably connected, hence divisible. It follows that every torsion element of the group S_2/S_1 (i.e. a coset of S_1) contains a torsion element of S_2 and therefore an element of $S_2 \cap C$. Hence, we have an injective map from $Tor(S_2/S_1)$ into $S_2 \cap C/S_1 \cap C$ and hence $|Tor(S_2/S_1)| \leqslant [S_2 \cap C : S_1 \cap C]$.

If $[S_2 : S_1]$ is infinite then $dim S_1 < dim S_2$ and therefore S_2/S_1 is a definably compact group of positive dimension. It follows from [9] that $Tor(S_2/S_1)$ is infinite and therefore, by the inequality above, so is $[S_2 \cap C : S_1 \cap C]$. If S_2/S_1 is finite then all its elements are torsion and therefore, by the same inequality we have

$$S_2/S_1 = Tor(S_2/S_1) = [S_2 \cap C : S_1 \cap C]$$

If S_1 is not definably connected then we apply the above argument first to $[S_2 : S_1^0]$ and $[S_1 : S_1^0]$ and then conclude the result for $[S_2 : S_1]$.

(2) Since A^{00} is divisible we only need, by (*) above, to see that Clause (i) holds for $Tor(A) + A^{00}$.

Let $S \subseteq A^{n+k}$ be a $\mathcal{M}$-definable group, $\pi_2 : S \to A^k$ the projection map onto the second coordinate and let $S_1 = \pi_2(S)$. As we saw in [13], we have

$$S^{00} = (A^{n+k})^{00} \cap S = (A^{00})^{k+n} \cap S$$

and

$$\pi_2(S^{00}) = S_1^{00} = (A^{00})^k \cap S_1.$$

Assume now that $(a, c_1+c_2) \in S$ for $a \in A$, $c_1 \in Tor(A)$, $c_2 \in A^{00}$. If $mc_1 = 0$ then $(ma, mc_2) \in S$, with $mc_2 \in S_1^{00}$. Because $S_1^{00} = \pi_2(S^{00})$ there exists $e_1 \in (A^{00})^n$ such that $(e_1, mc_2) \in S$. Moreover, because A^{00} is divisible, we have $e_1 = me$ for some $e \in (A^{00})^n$. It follows that $(ma, mc_2) - (me, mc_2) = (ma - me, 0) \in S$.

If we let k be the index of $S(A^n, 0)^0$ in $S(A^n, 0)$ then $(kma - kme, 0) \in S(A^n, 0)^0$ and there exists $(d, 0) \in S$ such that $kmd = kma - kme$. In particular, $(a - e) - d \in Tor(A)^n$. We now have $(a, c_1 + c_2) - (d, 0) = (a - d, c_1 + c_2) \in S$, with $a - d \in (Tor(A)^n + A^{00})^n$, as needed. □

By considering the special case of a compact real Lie group definable in the o-minimal structure $\mathbb{R}_{an}$ (or by a direct modified version of the above proof) we also have:

Lemma 7.4. *Let B be a connected, compact abelian real Lie group and let $\mathbf{B}_{an}$ be the expansion of $(B, +)$ by adding a predicate for every compact Lie subgroup of B^n.*

Then

$$\mathbf{Tor}(\mathbf{B})_{an} \prec \mathbf{B}_{an}.$$

We can now state the main result in the abelian case:

Theorem 7.5. *Let A be a definably compact, definably connected abelian group in a sufficiently saturated o-minimal expansion $\mathcal{M}$ of an ordered group. Endow $B = A/A^{00}$ with an L_{ab}-structure by interpreting R_S, for every 0-definable subgroup $S \subseteq A^n$, as the group $\pi(S) \subseteq B^n$ (where $\pi : A \to A/A^{00}$ is the projection map). Let $\mathbf{B}_{ab}$ be the induced structure on B. Then,*

1. *$\mathbf{B}_{ab} \equiv \mathbf{A}_{ab}$. Moreover, there is an L_{ab}-elementary embedding $\sigma : B \to A$ which is a section for π.*
2. *The structure $\mathbf{B}_{ab}$ is a reduct of $\mathbf{B}_{an}$ above.*

Proof. (1) We start with the structure $T = (\mathbf{Tor}(\mathbf{A}) + \mathbf{A^{00}})_{ab}$, which, by 7.3 (2), is an elementary substructure of $\mathbf{A}_{ab}$. By 7.2(3), there exists a homomorphic retract $\phi : A \to T$ and if we let $A_1 = ker\phi$ then, by 7.3(1), the structure $\mathbf{C}_{ab} = (Tor(A) + A_1)_{ab}$ is also elementary in $\mathbf{A}_{ab}$.

It is left to see that the restriction of π to $\mathbf{C}_{ab}$ induces an isomorphism of $\mathbf{C}_{ab}$ and $\mathbf{B}_{ab}$.

Let $S \subseteq A^n$ be an $\mathcal{M}$-definable group and $c \in C^n$. We need to see that $\pi(c) \in \pi(S)$ if and only if $c \in S$. Write $c = a + a_1$ for $a \in Tor(A)^n, a_1 \in A_1^n$, and assume that $\pi(a + a_1) \in \pi(S)$. It follows ($A^{00} \in ker\pi$) that for some $b \in (A^{00})^n$ we have $a + b + a_1 \in S$. Because A_1 is the kernel of the retract ϕ, we have $a + b \in S$. Because a is a torsion element, there exists an m such that

$$ma + mb = mb \in S \cap (A^{00})^n = S^{00}.$$

Because S^{00} is divisible, there exists $b_1 \in S^{00}$ such that $mb_1 = mb$ and therefore $b - b_1$ is a torsion element. However, $b - b_1$ belongs to the torsion-free group $(A^{00})^n$, hence $b = b_1$ and $b \in S$. We can therefore conclude that $c = a + a_1 \in S$, and therefore $\phi|C$ is an isomorphism of $\mathbf{C}_{ab}$ and $\mathbf{B}_{ab}$. The inverse map $\sigma : B \to C$ is an elementary embedding of $\mathbf{B}_{ab}$ into $\mathbf{A}_{ab}$.

(2) The image under the projection map of every definable set in A^n is closed in the Euclidean topology on $(A/A^{00})^n$. Since every closed subgroup of Lie group is itself a Lie subgroup, it follows that for every definable $S \leqslant A^n$, $\pi(S)$ is a Lie subgroup of B^n. Therefore, $\mathbf{B}_{ab}$ is a reduct of $\mathbf{B}_{an}$. □

Remark. Note that for all the results above we did not require the full theorem of Edmundo-Otero about the structure of $Tor(A)$ for a definably compact abelian group A. We only needed the weaker statement that every definably compact infinite group has infinitely many torsion elements. However, without the stronger result we will not be able to conclude that $\dim(A/A^{00}) = \dim A$.

Remark. It is not hard to see that $\mathbf{B}_{ab}$ is ω-saturated, hence it and $\mathbf{A}_{ab}$ will actually be $L_{\infty,\omega}$-equivalent, improving 7.5(1).

7.2. **The general case. Proof of Theorem 7.1**: By 6.4 and our earlier analysis, the group $H = [G, G]$ is definable, semisimple, $Z(G) \cap H$ is finite. We have $G \simeq (Z(G) \times H)/F$, for a finite central subgroup $F \subseteq Z(G) \times H$. Because G is definably

connected, H is definably connected as well. Hence, we also have $G \simeq (Z(G)^0 \times H)/F_1$, for a finite central subgroup F_1. We write $A = Z(G)^0$.

For any definably compact, definably connected K, we denote by $\widehat{K}$ the group K/K^{00}. For $a \in K$, we let $\hat{a} = \pi(a) \in \widehat{K}$.

We prove Theorem 7.1 in several steps.

Claim I If A is abelian and definably connected then for every finite subgroup $A_1 \subseteq A$,

$$\langle A, \cdot, \{a : a \in A_1\}\rangle \equiv \langle \widehat{A}, \cdot, \{\hat{a} : a \in A_1\}\rangle.$$

Moreover, there exists an elementary section $\sigma_A : \widehat{A} \to A$ with $\sigma_A(\hat{a}) = a$ for every $a \in A_1$.

Proof. Because A_1 is $\mathcal{M}$-definable, this is almost immediate from 7.5. We only need to notice that since all elements of A_1 are torsion elements, the elementary embedding of $\widehat{A}$ into A necessarily sends every $\hat{a} \in \widehat{A}$, $a \in A_1$, to the element a.

Claim II Given H definably connected, definably compact and semisimple, for every finite central subgroup $H_1 \subseteq H$,

$$\langle H, \cdot, \{a : a \in H_1\}\rangle \equiv \langle \widehat{H}, \cdot, \{\hat{a} : a \in H_1\}\rangle.$$

Moreover, there exists an elementary embedding $\sigma_H : \widehat{H} \to H$ which is a section for π, such that $\sigma_H(\hat{h}) = h$ for every $h \in H_1$.

Proof. By 4.4, we may assume that H is a semialgebraic group definable over the real algebraic numbers. Hence, every 0-definable set contains an element from $dcl(\emptyset)$, and moreover every finite group is 0-definable. We thus may assume that H is semialgebraic and definable over the real algebraic numbers.

In this case (see [13] for details), there exists an elementary embedding $\sigma_H : \hat{H} \to H$ which is also a section for $\pi : H \to \hat{H}$, and in particular, $H = H^{00} \rtimes \sigma_H(\hat{H})$. Because σ_H is elementary, for every $\hat{h} \in \widehat{H}_1$, the element $\sigma_H(\hat{h})$ is a torsion element of H and we have $\sigma_H(\hat{h})\hat{h}^{-1} \in H^{00}$. However H^{00} is torsion-free (see [2, Theorem 4.6]) and therefore $\sigma_H(\hat{h}) = h$.

Claim III If $G = (A \times H)/F$, for A definably connected, definably compact abelian, H definably connected, definably compact semisimple and F a finite central subgroup of $A \times H$, then $G \equiv \widehat{G}$. Moreover, there exists an elementary embedding $\sigma_G : \widehat{G} \to G$ which is a section for π.

Proof. Since F is finite it is contained in $A_1 \times H_1$ for some finite groups $A_1 \subseteq A$ and $H_1 \subseteq H$. It is easy to see that $\widehat{A \times H} \simeq \widehat{A} \times \widehat{H}$. By step I and step II, we have an elementary section:

$$\sigma : \langle \widehat{A} \sqcup \widehat{H}, \{\hat{a} \in \widehat{A_1}\}, \{\hat{h} \in \widehat{H_1}\}\rangle \to \langle A \sqcup H, \{a \in A_1\}, \{h \in H_1\}\rangle$$

sending each $\hat{a}$ and $\hat{h}$ to a and h respectively. It follows that we have elementary section

$$\sigma_1 : \langle \widehat{A} \times \widehat{H}, \cdot, \{(\hat{a}, \hat{h}) \in \widehat{A_1} \times \widehat{H_1}\}\rangle \to \langle A \times H, \cdot, \{(a, h) \in A_1 \times H_1\}\rangle,$$

and hence also

$$\sigma_2 : \langle \widehat{A} \times \widehat{H}, \cdot, \{\hat{g} \in \widehat{F_1}\}\rangle \simeq \langle \widehat{A \times H}, \cdot, \{\hat{g} \in \widehat{F_1}\}\rangle \to \langle A \times H, \cdot, \{g \in F_1\}\rangle.$$

This last section sends each $\hat{g} \in \hat{F}$ to $g \in F$.

In order to complete the proof of Claim III, it is therefore sufficient to prove the following general fact (with K now playing the role of $A \times H$):

Fact Let K be a definably connected, definably compact group and $F \subseteq K$ a finite central subgroup. Assume that $\sigma_K : \widehat{K} \to K$ is an elementary section of $\pi_K : K \to \widehat{K}$ which, for every $g \in F$ sends $\hat{g} \in \widehat{F}$ to $g \in F$. Then the map $\sigma_{K/F}$ which sends the element $(gF)(K/F)^{00}$ of $\widehat{G/F}$ to $(\sigma_K(gK^{00}))F \in K/F$ is an elementary section for $\pi_{K/F} : K/F \to \widehat{K/F}$.

Proof. It is easy to see that the map $\sigma : \widehat{K}/\widehat{F} \to K/F$ which sends $(gK^{00})\widehat{F}$ to $\sigma_K(gK^{00})F$ is elementary. It is also not hard to see that the map $\sigma' : (gF)(K/F)^{00}) \mapsto (gK^{00})\widehat{F}$ is an isomorphism of $\widehat{K/F}$ and $\widehat{K}/\widehat{F}$ (we use here the fact that the projection map $\pi_F : K \to K/F$ sends K^{00} onto $(K/F)^{00}$).

The composition of σ and σ' gives the desired $\sigma_{K/F}$.

8. Finite extensions of o-minimal groups

In this section we consider finite (but not necessarily central) extensions of arbitrary groups definable in o-minimal structure. Finite extensions of groups in o-minimal structures were studied by Edmundo, Jones and Peatfield in [8]. The following was shown there, using universal covers: If G is a definable, definably connected group in an o-minimal structure $\mathcal{M}$ expanding a real closed field, and if $\pi : \tilde{G} \to G$ is any finite definable extension of G, defined possibly in an o-minimal expansion $\mathcal{N}$ of $\mathcal{M}$, then $\tilde{G}$ is definably isomorphic in $\mathcal{N}$ to a group definable in $\mathcal{M}$.

Indeed, the above result is not stated as such but can be read off the proof of Proposition 3.2 in [8]. This implies for example that if G is semialgebraic then so is every finite extension of G which is definable in $\mathcal{N}$.

In this section we give two different proofs for similar results about the interpretability of finite extensions of definable groups and arbitrary topological covers of definable groups over the reals. Although the two main results, Theorem 8.3 and Theorem 8.4 overlap in the case of finite covers, the assumptions and techniques are different so we include both.

We first need the following fact about the structure of arbitrary definably connected groups in o-minimal structures. For G a group and $n \in \mathbb{N}$ we let $\sigma_n : G \to G$ be the map $\sigma_n(g) = g^n$.

Lemma 8.1. *Let G be a definably connected group in an o-minimal structure. Then,*
(i) The group $G/[G,G]$ is divisible, namely, for every $n \geqslant 1$, $\sigma_n(G)[G,G] = G$. In fact, there exists $k \in \mathbb{N}$ such that $\sigma_n(G)[G,G]_k = G$.
(ii) For every n, let $\langle \sigma_n(G)\rangle$ be the group generated by all elements g^n, $g \in G$. Then $G = \langle \sigma_n(G)\rangle$. In fact, there is $k \in \mathbb{N}$ such that $G = \sigma_n(G) \cdots \sigma_n(G)$ (k-times).

Proof. (i) We use induction on $\dim G$. If $\dim G = 1$ then G is abelian and therefore divisible. We consider the general case.

Assume first that G has an infinite definable normal abelian subgroup A. By induction, G/A satisfies the lemma and therefore, for every $n \in \mathbb{N}$, $A\sigma_n(G)[G,G] = G$. Because A is divisible, it is contained in $\sigma_n(G)$ and therefore $\sigma_n(G)[G,G] = G$, as needed.

If G has no infinite definable normal abelian subgroup then G is semisimple and therefore, by Claim 3.1, we have $[G,G] = G$.

For the last clause of (i), we may work in a sufficiently saturated structure, where the existence of such a k is clear. Once proved there, the same k works for G in any structure.

(ii) As before we may work in a sufficiently saturated structure. For G abelian the result is clear since it is divisible.

Assume that G has an infinite definable normal abelian subgroup A. By induction on dimension we have $G/A = \langle \sigma_n(G/A) \rangle$, which implies that $G = A\langle \sigma_n(G) \rangle$. However, since A is divisible it is contained already in $\sigma_n(G)$ and hence $G = \langle \sigma_n(G) \rangle$.

If G has no infinite definable normal abelian subgroup then it is semisimple. Let us see why the theorem is indeed true in this case.

We first assume that G is definably simple. If G is not definably compact then it is abstractly simple (see 1.1(5)). The group $\langle \sigma_n(G) \rangle$ is clearly invariant under all automorphisms of G hence normal, so $G = \langle \sigma_n(G) \rangle$. If G is definably compact, then by 1.1 it is elementarily equivalent to a simple compact real Lie group H. By simplicity, $H = \langle \sigma_n(H) \rangle = \bigcup_{k=1}^{\infty} \sigma_n(H) \cdots \sigma_n(H)$ (k-times). It follows from compactness that for some k we have $H = \sigma_n(H) \cdots \sigma_n(H)$ (k-times). This implies that the same is true for G.

If G is semisimple then $Z(G)$ is finite and we have $G/Z(G) = H_1 \times \cdots \times H_r$, for H_i definably simple. By the above, each H_i satisfies $H_i = \langle \sigma_n(H_i) \rangle$, and hence we have $G = Z(G)\langle \sigma_n(G) \rangle$, so $\langle \sigma_n(G) \rangle$ has finite index in G. However, $\langle \sigma_n(G) \rangle$ is a countable union of definable sets and therefore it follows that G is a countable union of such sets. Because of saturation, this implies that $\langle \sigma_n(G) \rangle$ is actually definable (in finitely many steps) and by the definable connectedness of G we have $G = \langle \sigma_n(G) \rangle$. □

We prove:

Theorem 8.2. *Let $\mathcal{M}$ be an arbitrary structure, sufficiently saturated, and let $\mathcal{R}$ be a definable o-minimal structure in $\mathcal{M}$. Let G be an $\mathcal{R}$-definable group, $\tilde{G}$ an $\mathcal{M}$-definable group and let $\pi : \tilde{G} \to G$ be a $\mathcal{M}$-definable surjective homomorphism with finite kernel N.*

Then $\tilde{G}$ is internal to G in the reduct containing $\langle \tilde{G}, \cdot \rangle$, $\langle G, \cdot \rangle$, π and a predicate for G^0 the definably connected component of G (denote this reduct by $\mathcal{M}'$).

More precisely, $\tilde{G}$ is in the $\mathcal{M}'$-definable closure of G and a finite subset of $\tilde{G}$.

Proof. Let $\pi : \tilde{G} \to G$ be the extension map. We may assume that G is definably connected (in the sense of $\mathcal{R}$). Indeed, since N is finite, $\pi^{-1}(G^0)$ has finite index in $\tilde{G}$, so if $F \subseteq G$ is a finite set such that $G = FG^0$ then $\tilde{G}$ is in the $\mathcal{M}$-definable closure of $\pi^{-1}(G^0)$ and the finite set $\pi^{-1}(F)$. It follows that if $\pi^{-1}(G^0)$ is $\mathcal{M}'$-internal to G then so is $\tilde{G}$.

We may also assume that $\tilde{G}$ has no $\mathcal{M}'$-definable subgroups of finite index. Indeed, if $\tilde{G}_1 \subseteq \tilde{G}$ is definable of finite index then $\pi(\tilde{G}_1)$ has finite index in G, and therefore (we assume G is definably connected) $\pi(\tilde{G}_1) = G$. This in turn implies that N is not contained in $\tilde{G}_1$, and therefore $|N \cap \tilde{G}_1| < |N|$. Using induction on $|N|$ we could finish.

The assumption that $\tilde{G}$ has no $\mathcal{M}'$-definable subgroups of finite index implies that N is central in G. As we will show, under these assumptions, $\tilde{G}$ is in the definable closure of G and N.

Let $n = |N|$.

For $k \in \mathbb{N}$, let f_k be the term $f_k(x_1, \ldots, x_k) = x_1^n \cdots x_k^n$, and for a group H, let $f_{k,H} : H^k \to H$ be the evaluation of the term in H.

Let $\pi : \tilde{G}^k \to G^k$ be the projection map in each of the coordinates. Similarly to Claim 2.2, we claim that for $\bar{g}_1, \bar{g}_2 \in \tilde{G}^k$, if $\pi(\bar{g}_1) = \pi(\bar{g}_2)$ then $f_{k,\tilde{G}}(\bar{g}_1) = f_{k,\tilde{G}}(\bar{g}_2)$ (we use the fact that N is central and for every $h \in N$ we have $h^n = e$).

It now follows that there is a $\mathcal{M}'$-definable surjective map $h_k : G^k \to \sigma_n(\tilde{G}) \cdots \sigma_n(\tilde{G})$ (k-times) such that $f_{k,\tilde{G}}$ factors through π and h_k.

By Theorem 8.1, we may choose k such that $G = \sigma_n(G) \cdots \sigma_n(G)$ (k-times). Said differently, the map $f_{k,G} : G^k \to G$ is surjective. It easily follows that $\tilde{G} = N h_k(G^k)$. □

Theorem 8.3. *Let $\mathcal{M}$ be an o-minimal structure and assume that $1 \to N \to \tilde{G} \to G \to 1$ is an $\mathcal{M}$-definable extension with N finite and $\tilde{G}$ definably connected.*

Let $\mathbf{G}$ be some expansion of $\langle G, \cdot \rangle$ with property ρ.

Then $1 \to N \to \tilde{G} \to G \to 1$ is definably isomorphic in $\mathcal{M}$ to an a extension $1 \to N_1 \to \tilde{G}' \to G \to 1$ which is definable in $\mathbf{G}$ over parameters (with $h_G : G \to G$ the identity map).

The parameters name a bijection between a finite collection $\mathcal{W}$ of $\mathbf{G}$-definable sets and a finite subset of N. The collection $\mathcal{W}$ (but not necessarily each of its sets) is 0-definable in the pure group G.

Proof. Note that because $\tilde{G}$ is definably connected and N is normal and finite then it is necessarily central. The proof of this theorem is very similar to that of Theorem 2.1. Instead of products of k commutators (i.e. the function $F_{k,G}$) we use products of k-many n-powers of G (the function $f_{k,G}$ defined above), with $n = |N|$. Also, instead of k_n we use here the function $h_k : G^k \to \tilde{G}$ defined above and instead of the set $G(k)$ defined there we use the set

$$G_k = \{\bar{g} \in G^k : f_{k,G}(\bar{g}) = e\}$$

and its definably connected components.

Finally, instead of using the fact there that every element of the perfect G was a product of k commutators, we use Theorem 8.1 which implies that every element of G is a product of k nth-powers. The other details are identical to the proof of Theorem 2.1. □

8.1. **The real case.** Just like in case of Theorem 2.8, if one works over the field of real numbers then there is no need to assume that $\tilde{G}$ is definable and we obtain the following version of Theorem 8.3:

Theorem 8.4. *Let $\mathcal{M}$ be an o-minimal structure over the real numbers, G an $\mathcal{M}$-definable group and assume that $E : 1 \to N \to \tilde{G} \to G \to 1$ is a topological extension with N finite and $\tilde{G}$ connected.*

Let $\mathbf{G}$ be some expansion of $\langle G, \cdot \rangle$ with property ρ.

Then E is isomorphic as a topological extension, to an extension $E' : 1 \to N_1 \to \tilde{G} \to G \to 1$ definable in the structure $\mathbf{G}$*, (with the isomorphism being the identity on G).*

Proof. The arguments as to why this version of 8.4 is true are identical to those explaining the proof of 2.8. In both cases the only o-minimal facts that are being used apply to G (rather than $\tilde{G}$). □

We end this diversion into extensions of definable real Lie groups by considering topological covers and related central extensions. This is really an application of work by Edmundo ([6]) and Edmundo-Eleftheriou ([7]) on universal covers and local definability in an *o*-minimal setting, as well as work on definable fundamental groups by Berarducci and Otero ([3]). We include the material because we could not find it precisely stated in the literature. In any case thanks to Edmundo for his explanations to us of results implicit in his work, some of which we repeat in the proof below.

Let us now set up notation for Theorem 8.5 below. $\mathcal{M} = \langle \mathbb{R}, <, +, \cdot, \cdots \rangle$ will be an *o*-minimal expansion of the real field, and G a definably connected group definable in $\mathcal{M}$ (so G is what we have called a definable real Lie group). $\tilde{G}$ will be the topological universal cover of G (also a connected real Lie group) and Γ denotes the kernel of $\tilde{G} \to G$, namely the fundamental group $\pi_1(G)$ of G. So Γ is a central discrete closed subgroup of $\tilde{G}$. If $f : \Gamma \to A$ is a homomorphism from Γ into an abelian group A, we form as usual the group $\tilde{G}_A = \tilde{G} \times_\Gamma A$, and we have a central extension $1 \to A \to \tilde{G}_A \to G \to 1$ of G (as abstract groups). We will refer to locally definable groups, for which the reader can consult [7], although we give an explanation inside the proof.

Theorem 8.5. *(i) $\tilde{G}$ and the covering homomorphism can be realized, even topologically, as a locally definable group and homomorphism in $\mathcal{M}$.*
(ii) $\tilde{G}_A$ with its group structure, the extension $1 \to A \to \tilde{G}_A \to G \to 1$, together with a section $G \to \tilde{G}_A$, can be interpreted with parameters in the two sorted structure consisting of $\mathcal{M}$ and $\langle A, + \rangle$.

Proof. We will be brief. But note first that taking $A = \Gamma$ and h the identity, (ii) says that $\tilde{G}$ can be interpreted in the two sorted structure consisting of $\mathcal{M}$ and $\langle \Gamma, + \rangle$.

Recall first that for an arbitrary central group extension $E : 1 \to A \to H \to_\pi G \to 1$, if $s : G \to H$ is a section for π, and $h_s(x, y) = s(xy)^{-1}s(x)s(y)$, then h_s (which is called a 2 co-cycle) is a map from $G \times G \to A$ and the group H is isomorphic to the group H' whose underlying set is $G \times A$ and whose group operation is given by $(x, a) \cdot (y, b) = (xy, h_s(x, y) + a + b)$ (we can write the second coordinate additively because A is abelian). Moreover $\pi' : H' \to G$ is the usual projection, the embedding of A into H' is given by $a \mapsto (1, -a - h_s(1, 1))$, and the section s is just $x \mapsto (x, 0)$. Hence, in order to recover E we only need to find such a definable co-cycle h_s.

We now prove part (i). The statement of (i) is that there exists in $\mathcal{M}$ a locally definable group $\mathcal{U} = \bigcup_{i \in I} X_i$ (i.e. a bounded directed union of definable sets) with a locally definable group operation (i.e. it is definable when restricted to each $X_i \times X_j$), and a locally definable surjective homomorphism $w : \mathcal{U} \to G$, and moreover the group $\mathcal{U}$ with its topology as a locally definable group in $\mathcal{M}$ is precisely the universal covering of G. The reasoning is as follows: Choose $w : \mathcal{U} \to G$ to be the universal locally definable cover of G as described in [7]. $\mathcal{U}$ has of course a topology

as a locally definable group, and as the underlying set of $\mathcal{M}$ is $\mathbb{R}$, it will be locally Euclidean, connected, and a topological cover of G. So w induces an embedding w_* of the topological fundamental group $\pi_1(\mathcal{U})$ of $\mathcal{U}$ into $\pi_1(G)$. We will point out that $\pi_1(\mathcal{U}) = 0$, whereby $\mathcal{U}$ *will be* the universal cover $\tilde{G}$ of G. Let $c \in \pi_1(\mathcal{U})$. So $w_*(c) \in \pi_1(G)$. By [3], $\pi_1(G) = \pi_1^{def}(G)$ (the definable fundamental group of G), whereby $w_*(c)$ is represented by a definable path γ (beginning and ending at the identity). By Lemma 2.7(1) of [7], γ lifts to a definable path γ' in $\mathcal{U}$ starting at the identity. On general topological grounds, γ' is a loop, and represents c. But $\pi_{def}(\mathcal{U})$ is trivial, whereby γ' is definably homotopic to the identity. Thus $c = 0$.

For (ii) let us first prove the special case that $\mathcal{U}$ is definable in $\langle \mathcal{M}; \Gamma \rangle$. Because I is bounded, there is an $i_0 \in I$ such that X_{i_0} projects onto G. Because of definable choice, there is an $\mathcal{M}$-definable $Y \subseteq X_{i_0}$ and an $\mathcal{M}$-definable $s : G \to Y$ which is a section for w. Moreover, because of local definability, there is $j \in I$ such that $Y \cdot Y \cdot Y^{-1} \subseteq X_j$, hence the associated 2 co-cycle $h_s : G \times G \to X_j$ is also $\mathcal{M}$-definable and its image is contained in X_j. Note that since $w|X_j : X_j \to G$ is definable it follows that the image of h_s in Γ must be finite (otherwise the kernel of π in X_j will be an infinite definable discrete set). Finally, as mentioned above, given the co-cycle h_s we can recover a definable covering $1 \to \Gamma' \to \mathcal{U}' \to G \to 1$ in $\langle \mathcal{M}; \langle \Gamma, + \rangle \rangle$ which is isomorphic to the original one. Because Γ is a bounded set (independently of the model $\mathcal{M}$) the set $\mathcal{U}' = \Gamma \times G$ can be written as a directed union of definable sets. Since h_s is an $\mathcal{M}$-definable map the group structure on $\mathcal{U}'$ is locally definable. Finally, the isomorphism $(x, a) \mapsto x \cdot a$ from $\mathcal{U}'$ to $\mathcal{U}$ is locally definable as well and therefore a homeomorphism.

We now consider the general case of (ii). By what has been done so far we may identify $\tilde{G}$ with $\mathcal{U}$. Let $h_s : G \times G \to \Gamma$ be the 2-cocycle from the previous paragraph. Define $h' : G \times G \to A$ to be $f \circ h_s$ (where recall f is the given homomorphism of Γ into A). Then h' is precisely the 2-cocycle determining $\tilde{G}_A$. As h_s was definable in $\mathcal{M}$ with finite image in Γ, it follows that the group operation on $G \times A$ given in the first paragraph of the proof is definable in the two sorted structure consisting of $\mathcal{M}$ and $\langle A, + \rangle$. $\square$

Let us remark in closing this section that Theorem 8.5 gives an interesting twist on certain covering structures considered by Zilber, such as the two sorted structure M_0, say, consisting of $\langle \mathbb{C}, + \rangle$ in one sort, $\langle \mathbb{C}, +, \cdot \rangle$ in the other sort and the complex exponential map *exp* going from the first sort to the second. The kernel of *exp* can be identified with the (definable in M_0) subgroup $\mathbb{Z}$ of the first sort. It is easy to see that the full structure M_0 cannot be interpreted in the reduct consisting of the sorts $\langle \mathbb{Z}, + \rangle$ and $\langle \mathbb{C}, +, \cdot \rangle$. But Theorem 8.5 says that M_0 can be so interpreted if we add a predicate for $\mathbb{R}$ to the second sort.

We should also mention the Ph.D. thesis of Misha Gavrilovich on the model theory of the universal covering spaces of complex algebraic varieties, which contains ideas and constructions related to ours above.

9. Groups which are not definably connected

In this section we prove an analogue of Theorem 7.1 for definably compact groups which are not assumed to be definably connected.

We assume that $\mathcal{M}$ is a sufficiently saturated o-minimal structure expanding a real closed field. We still use $\widehat{G}$ to denote G/G^{00}. Here are some preliminaries:

Claim 9.1. *(i) Let G be any group, and H a subgroup of finite index. Then there are $g_1, .., g_n \in G$, and $h_1, .., h_m \in H$ such that the structures $(G, \cdot, H, g_1, .., g_n)$ and $(H, \cdot, a_1, .., a_n, h_1, .., h_m)$ are bi-interpretable, where a_i is conjugation by g_i.*
(ii) In the special case when G is definable in an o-minimal structure and $H = G^0$, then G^0 is definable in $\langle G, \cdot \rangle$ so in fact $(G, \cdot, g_i)_i$ and $(G^0, \cdot, h_j, a_i)_{j,i}$ are bi-interpretable.

Proof. (i) is straightforward: let the g_i be representatives of cosets of H in G, and for each i, k let $g_i g_k = c_{ik} g_r$ for suitable r and $c_{ik} \in H$. Let the h_j's be the c_{ik}'s. Details are left to the reader.
(ii) In the notation of lemma 8.1, there is an n such that $\sigma_n(G) \subseteq G^0$ and therefore, by the same lemma, there is a k such that $G^0 = \sigma_n(G) \cdots \sigma_n(G)$ (k-times). This implies that G^0 is definable (without parameters) in G. □

By the above, in order to understand an arbitrary definable group G we need to understand G^0 together with finitely many definable automorphisms.

By 6.4, every definably compact, definably connected group G is the almost direct product of the semisimple group $[G, G]$ and $Z(G)^0$.

Clearly, every definable automorphism of G leaves invariant both $[G, G]$ and $Z(G)^0$, so we need to understand each of the two groups, together with finitely many definable automorphisms.

Theorem 7.5 allows us to treat definable automorphisms of a definable abelian group A (by viewing their graph as a subgroup of $A \times A$). Hence, we now need to examine definable automorphisms of definable semisimple groups.

Claim 9.2. *Let $\mathcal{M}$ be an o-minimal expansion of a real closed field R. If G is an R-semialgebraic, definably connected, definably compact semisimple group, then every definable automorphism of G is R-semialgebraic.*

Proof. Assume first that G is definably simple and $f : G \to G$ is a definable automorphism. Because G is definably compact it is bi-interpretable with a real closed field R_1. Therefore, by [19, Proposition 4.8], (a Borel-Tits-style result), $f = g \circ h$, where g is an R-semialgebraic automorphism of G and h is induced by an automorphism σ of the semialgebraic field R_1. The proof of Proposition 4.8 cited above shows that σ is definable, if f is definable. So σ is definable and thus the identity. So f is semialgebraic, proving the claim in the special case.

Assume now that G is semisimple and centerless. Hence, by 1.1, G is definably isomorphic in R to $H_1 \times \cdots \times H_n$, where each H_i is a linear semialgebraic group, defined over the real algebraic numbers $R_{alg} \subseteq R$. Without loss of generality, $G = H_1 \times \cdots \times H_n$, and we consider each H_i as a subgroup of G. For each $i = 1, \ldots, k$ we denote by $\pi_i : G \to H_i$ the projection map. Without loss of generality, $dim H_n \leqslant \dim H_i$ for every $i \leqslant n$.

Let $f : G \to G$ be a definable automorphism. It is clearly sufficient to see that $f : H_i \to G$ is semialgebraic for every $i = 1, \ldots, n$. We prove that every definable embedding $\phi : H_i \to G$ is semialgebraic. We first claim that $\pi_n \phi(H_i) = \{e\}$ or $\pi_n \phi(H_i) = H_n$. Indeed, if $\pi_n \phi(H_i) \neq H_n$ then by the dimension assumption, $ker \pi_n \phi \neq \{e\}$, which by simplicity implies that $\pi_n \phi(H_i) = \{e\}$).

Now, if $\pi_n\phi(H_i) = \{e\}$, we can reduce the problem by induction to a group G of smaller dimension. If $\pi_n\phi(H_i) = H_n$ then the map $\pi_n\phi$ is an isomorphism of H_i and H_n and therefore, similarly to the proof for the definably simple case, $\phi|H_i$ is semialgebraic.

Let G be an arbitrary definably connected semisimple group and let $H = G/Z(G)$. By 4.4, G is bi-interpretable with $G/Z(G)$ over parameters in $G/Z(G)$ and $Z(G)$. Moreover, by the last clause of Theorem 2.1, these parameters are 0-definable in $\langle G, \cdot\rangle$. Hence, there is a definable group G_1 in the structure $\langle G/Z(G), Z(G)\rangle$, and a G-definable isomorphism $\sigma : G \to G_1$, where G_1 and σ are invariant under every automorphism of the group G.

Now, let $f : G \to G$ be a definable automorphism in $\mathcal{M}$. The map f induces an automorphism f_1 of $G/Z(G)$ and since $G/Z(G)$ is centerless ($Z(G)$ is finite) it follows from the centerless case that f_1 is R-semialgebraic. As we just pointed out, f leaves G_1 invariant and hence $f_1|G_1$ is an R-semialgebraic automorphism of G_1. Composing with the f-invariant σ we see that f itself is also R-semialgebraic. □

The following lemma is general.

Lemma 9.3. *Let* $\mathbf{G}$ *be an arbitrary group,* $A \subseteq G$ *a central subgroup of* G*, such that for some number* k*, every element of* G/A *equals the product of* k *commutators from* G/A.

Let $f : G \to G$ *be a group automorphism of* G *such that* $f(A) = A$*. Then* f *is definable in the structure* $\mathbf{G} = \langle G, , \cdot, A, f|A, f|G/A\rangle$.

Proof. We use Beth definability theorem: Namely, we take $\langle \tilde{G}, \tilde{A}, \tilde{f}|A, \tilde{f}|G/A\rangle$ elementarily equivalent to $\mathbf{G}$ and show that there is unique automorphism $g : \tilde{G} \to \tilde{G}$ leaving A invariant such that $g|A = \tilde{f}|A$ and $g|G/A = \tilde{f}|G/A$. Note that the assumption on G implies that $\tilde{G}$ is still the product of $\tilde{A}$ and $[\tilde{G}, \tilde{G}]$ (this is true in G and because $[G/A, G/A]$ is generated in finitely many steps, it becomes a first order statement true in $\tilde{G}$ as well).

Assume that we have $g, h : \tilde{G} \to \tilde{G}$ automorphisms as above and consider gh^{-1}. Then $gh^{-1}|\tilde{A} = id$ and $gh^{-1}|\tilde{G}/\tilde{A} = id$. We may therefore assume that $g|\tilde{A}$ and $g|\tilde{G}/\tilde{A}$ are the identity maps and aim to show that $g = id$.

Because $g|\tilde{G}/\tilde{A} = id$, for every $x \in \tilde{G}$, we have $x^{-1}g(x) \in \tilde{A}$ and hence there exists a function $a : \tilde{G} \to \tilde{A}$ such that $g(x) = xa(x)$. We claim that a is a group homomorphism: For $x, y \in \tilde{G}$ we have

$$xya(xy) = g(xy) = g(x)g(y) = xa(x)ya(y) = xya(x)a(y)$$

(because $a(x), a(y)$ are central elements). It follows that $a(xy) = a(x)a(y)$.

For every $x \in A$ we have $g(x) = x$, hence $a(x) = e$. Also, if $b = xyx^{-1}y^{-1}$ is a commutator in $\tilde{G}$ then $g(b) = ba(xyx^{-1}y^{-1}) = b$, hence $a(b) = e$. But then $ker(a)$ contains both A and the commutator subgroup of $\tilde{G}$. Because $\tilde{G}$ is generated by these two groups, $a(x) = e$ for all $x \in \tilde{G}$ and therefore $g = id$. □

Theorem 9.4. *If* G *is a definably compact group in an o-minimal expansion* $\mathcal{M}$ *of an ordered group then it is elementarily equivalent to a definably compact, semialgebraic (over parameters) group* H *over a real closed field, with* $\dim H = \dim G$.

Proof. As we saw above, G is bi-interpretable, over parameters from G^0, with G^0 together with the action of finitely many automorphisms $f_1, \ldots, f_k$. For simplicity, we denote G^0 by H and $Z(H)^0$ by A.

We also saw, in 9.3, that the structure

$$\langle H, \cdot, \{f_1, \dots, f_k\}, \{c_1, \dots, c_r\}\rangle$$

(for constants $c_1, \dots, c_r \in H$) is definable in

$$\langle H, \cdot, A, \{f_i|A : i = 1, \dots, k\}, \{f_i|H/A : i = 1, \dots, k\}, \{c_1, \dots, c_r\}\rangle.$$

We denote each $f_i|H/A$ by g_i and each $f_i|A$ by h_i.

By Theorem 6.1, the group H is definable, over parameters, in the two-sorted structure $\langle H/A, A\rangle$. Putting it all together, we see that G is definable, over parameters, in $\langle H/A, \{g_i : i = 1, \dots, k\}, A, \{h_i : i = 1, \dots, k\}\rangle$. (where H/A and A are endowed with their group structure).

By 4.4 the semisimple group H/A is definably isomorphic to a semialgebraic group G_0 over $R_{alg} \subseteq R$, for a real closed field R and by 9.2, each g_i is sent by this isomorphism to an R-semialgebraic automorphism of G_0, possibly defined over parameters.

The structure $\langle A, +, \{h_1, \dots, h_k\}\rangle$ is clearly a reduct of the structure $\mathbf{A}_{ab}$ considered in Theorem 7.5 (since every automorphism of G gives rise to a subgroup of $G \times G$). Therefore, it is elementarily equivalent to an expansion of a connected, compact, abelian real Lie group $\widehat{A}$ (with $\dim \widehat{A} = \dim A$), by Lie group automorphisms $\widehat{h}_1, \dots, \widehat{h}_k$. Finally, $\widehat{A}$, as a compact Lie group, is isomorphic to a real algebraic linear group L. This isomorphism takes each $\widehat{h}_i$ to a a Lie subgroup of L^2, which itself must be semialgebraic (indeed, this last fact follows for example from [18, 3.3], applied to the o-minimal structure $\mathbb{R}_{an}$, in which every definable compact linear group is definable).

Hence, by going to a sufficiently saturated real closed field $\widetilde{R}$, we can find constants $d_1, \dots, d_k \in \widetilde{R}$ such that

$$\mathcal{M}_1 = \langle A, \{h_i : i = 1, \dots, k\}, H/A, \{g_i : i = 1, \dots, k\}, \{c_i := i = 1, \dots, r\}\rangle$$

is elementarily equivalent to

$$\mathcal{M}_2 = \langle L(\widetilde{R}), \{\widehat{h}_i : i = 1, \dots, k\}, G_0(\widetilde{R}), \{\widehat{g}_i : i = 1 \dots, k\}, \{d_i : i = 1, \dots, r\}\rangle,$$

with G_0, L, and the automorphisms $\widehat{g}_i, \widehat{h}_i$ all semialgebraic.

Because G is definable over parameters in $\mathcal{M}_1$, it is elementarily equivalent to a group definable (over parameters) in $\mathcal{M}_2$, and this last group must be semialgebraic. □

Remark. By Lemma 11.3, the parameters in A can be realized as real algebraic elements in an elementarily equivalent real group. However, we don't know how to do the same for the parameters in H/A.

10. Compact domination for definably compact groups

Here we give another application of Corollary 6.4. The "compact domination conjecture" for *definably compact*, definably connected groups in (saturated) *o*-minimal expansions of real closed fields, was introduced in [13]. The conjecture says that G is dominated by G/G^{00} equipped with its Haar measure. Namely, writing $\pi : G \to G/G^{00}$ for the canonical surjective homomorphism, for any definable subset X of G the set of $c \in G/G^{00}$ such that $\pi^{-1}(c)$ intersects both X and its complement, has Haar measure 0. We sometimes just say "G is compactly dominated".

The conjecture was proved in [13] for G with "very good reduction", and by part (ii) of Theorem 4.4 of the current paper, this is the case for semisimple definably connected groups. In [14] compact domination was proved for G commutative. With 6.4 we know that arbitrary G (definably compact, definably connected) almost splits into its semisimple and abelian parts, and one would expect that this makes it easy to deduce compact domination of G from the two special cases.

We first prove:

Theorem 10.1. *Every definably connected, definably compact group in $\mathcal{M}$ a sufficiently saturated expansion of a real closed field is compactly dominated.*

Proof. We first prove the result for a group $G \times H$, with G commutative and H semisimple. It is sufficient to prove the result for each definable set separately so we may assume that the language is countable. We fix $\mathcal{M}_0 \subseteq \mathcal{M}$ a countable model. We let

$$\pi : G \times H \to G/G^{00} \times H/H^{00},$$

with $\pi = (\pi_1, \pi_2)$ and $\pi_1 : G \to G/G^{00}$, $\pi_2 : H \to H/H^{00}$.

The following observation is true in greater generality (for any type-definable equivalence relation), but we only observe it in the o-minimal setting: If K is a definably compact group in $\mathcal{M}$, definable over M_0 and $a_1, a_2 \in K$ realize the same type over M_0 then the lie in the same K^{00}-coset.

Indeed, if σ is any automorphism of $\mathcal{M}$ then it induces a continuous (with respect to the logic topology) automorphism of K/K^{00} which fixes all the torsion points of K (since they belong to M_0). But $\pi(Tor(K))$ is dense in K/K^{00}, therefore σ induces the identity map on K/K^{00}. This is clearly sufficient.

Let $X \subseteq G \times H$ be a definable set over M_0 and assume, towards contradiction, that the set

$$B = \{(g', h') \in G/G^{00} \times H/H^{00} : \pi^{-1}(g', h') \cap X \neq \emptyset \,\&\, \pi^{-1}(g', h') \cap X^c \neq \emptyset\}$$

has positive measure.

Recall from [14] that $\mathcal{M}^*$ denotes the expansion of $\mathcal{M}$ by adjoining relations for all externally definable sets, and that the theory of this expansion is weakly o-minimal. From [13], Fin denotes the finite elements of $\mathcal{M}$ and Inf the infinitesimals, both definable in $\mathcal{M}^*$. Fin/Inf identifies with $\mathbb{R}$ and the structure on it that is induced from $\mathcal{M}^*$ is o-minimal. Moreover H/H^{00} is a definable subset of some $(Fin/Inf)^n$ and the logic topology on H/H^{00} coincides with its topology as a definable group in Fin/Inf. In particular any subset of H/H^{00} definable in $\mathcal{M}^*$ which has positive Haar measure, is of maximal o-minimal dimension, so has interior. The results in [14] give a similar picture for G/G^{00}. Namely G^{00} is definable in $\mathcal{M}^*$ and G/G^{00} is semi-o-minimal, namely lives in the product of finitely many definable o-minimal sets in $\mathcal{M}^*$. The two topologies (logic, o-minimal) again coincide, so definable (in $\mathcal{M}^*$) sets of positive Haar measure have interior.

The group $G/G^{00} \times H/H^{00}$, with all its induced $\mathcal{M}^*$ structure, is also semi-o-minimal and because the set B is definable in $\mathcal{M}^*$ there are open sets $U \subseteq G/G^{00}$ and $V \subseteq H/H^{00}$ with $U \times V \subseteq B$. We claim that there exists $g' \in U$ such that all elements of $\pi_1^{-1}(g')$ realize the same type in $\mathcal{M}$, over M_0. Indeed, because G is compactly dominated, for every M_0-definable subset X of G, the set of all $g' \in G/G^{00}$ such that $\pi_1^{-1}(g') \cap X \neq \emptyset$ and $\pi_1^{-1}(g') \cap X^c \neq \emptyset$ has Haar measure zero. So, after

removing countably many such sets (each of measure zero), the pre-image of every $g' \in U$ under π_1 is contained in a complete $\mathcal{M}$-type over M_0.

We fix one such $g' \in U$ as above, $g \in \pi_1^{-1}(g')$, and consider the set $X_g = \{h \in H : (g,h) \in X\}$. We claim that for every $h' \in V$, the sets $\pi_2^{-1}(h') \cap X_g$ and $\pi_2^{-1}(h') \cap X_g^c$ are both nonempty, contradicting the fact that H is compactly dominated. Indeed, if $h' \in V$ then, by assumption on B, there are $g_1, g_2 \in \pi_1^{-1}(g')$ and $h_1, h_2 \in \pi_2^{-1}(h')$ with $(g_1, h_1) \in X$ and $(g_2, h_2) \in X^c$. Because g_1, g_2 and g all realize the same type over M_0, there are $h_3, h_4 \in H$ conjugates over M_0 of h_1, h_2, respectively, with $(g, h_3) \in X$ and $(g, h_4) \in X^c$. By our earlier observation, h_3 and h_4 belong to the pre-image of h', so $\pi_2^{-1}(h') \cap X_g$ and $\pi_2^{-1}(h') \cap X_g^c$ are non-empty. Contradiction. We thus showed that $G \times H$ is compactly dominated.

The result for an arbitrary definably compact group follows from the special case using Corollary 6.4, noting that compact domination is preserved under quotients (using the fact that a definable surjective homomorphism $\sigma : G_1 \to G_2$ of definably compact groups sends G_1^{00} onto G_2^{00}).

□

11. Appendix: On Abelian groups

Since all groups here are abelian we write them additively. The first two lemmas are standard and we include them for completeness.

Lemma 11.1. *Let $B \subseteq A$ be two abelian divisible groups such that B contains all torsion elements of A and let A_0 be a subgroup of A. Then the following are equivalent:*
(i) A_0 is a maximal subgroup of A such that $A_0 \cap B = \{0\}$.
(ii) $A = A_0 \oplus B$ and A_0 is divisible.
(iii) $A = A_0 \oplus B$.

Proof. We only need to see that (i) implies (ii). Notice that the assumptions on B (with A divisible) are equivalent to: for every $n \geqslant 1$ and $a \in A$, if $na \in B$ then $a \in B$.

We first show:
Claim For every $n \geqslant 1$ and $a \in A$, if $na \in A_0 + B$ then $a \in A_0 + B$.

We use induction on n: The case $n = 1$ is obvious. For general n, if $na = a_0 + b$ for some $a_0, \in A_0, b \in B$, then, since A is divisible there exists $a' \in A$ such that $na' = a_0$. It follows that $n(a - a') \in B$ and therefore, by the above observation, $a - a' \in B$. It is therefore sufficient to prove that $a' \in A_0 + B$.

If $a' \notin A_0$ then, by the maximality of A_0, there exists $k \in \mathbb{Z}$, and $a_1 \in A_0$ such that $ka' + a_1 = b' \in B$ and $b' \neq 0$. We write $k = mn + \ell$, $0 \leqslant \ell < n$ and then we have $mna' + \ell a' + a_1 = b'$. If $\ell = 0$ then $mna' + a_1 = b' \neq 0$, which is impossible because $mna' + a_1 \in A_0$. It follows that $\ell \neq 0$ and we have $\ell a' = (-mna' - a_1) + b \in A_0 + B$. By induction, we have $a' \in A_0 + B$. End of Claim.

We can now prove (ii): For $a \in A$, if $a \notin A_0$ then, by the maximality of A_0, there exist $k \in \mathbb{Z}$ and $a_0 \in A_0$ such that $0 \neq ka + a_0 = b \in B$. By assumption on A_0, we have $k \neq 0$ (and $ka \in A_0 + B$). The last claim implies that $a \in A_0 + B$.

To see that A_0 is divisible, take $a_0 \in A_0$ and $n \geqslant 1$. Because A is divisible, we have $a_1 \in A_0, b \in B$, such that $na_1 + nb = a_0$. It follows that $nb = 0$ and then we also have $na_1 = a_0$. $\square$

Lemma 11.2. *Let A, B be two divisible abelian groups such that B has unbounded exponent. Assume that $\phi : B \to A$ is a group embedding, with $Tor(A) \subseteq \phi(B)$. Then ϕ is an elementary map.*

Proof. We assume that $B \subsetneq A$ and by moving to an elementary extension of $\langle A, +, B\rangle$ we may assume that B contains a torsion-free element. By 11.1, we can write $B = Tor(A) \oplus B_0$ for some torsion-free divisible subgroup B_0. Because B has unbounded exponent, $B_0 \neq \{0\}$. Applying again 11.1, we can write $A = Tor(A) \oplus B_0 \oplus A_0$ for $A_0 \neq \{0\}$. We now use the fact that $B_0 \prec (B_0 \oplus A_0)$ (as divisible torsion-free abelian groups) to conclude that $B \prec A$. $\square$

We also need the following claim on definable abelian groups in o-minimal structures.

Lemma 11.3. *Let $\mathcal{M}$ be an o-minimal expansion of an ordered group. If G is a definable abelian group (possibly not definably connected) and C is a finite subset of G then $\langle G, +, \{b :\in C\}\rangle$ (namely, we add a constant to every element of C) is elementarily equivalent to a real algebraic group of the same dimension, with finitely many real algebraic elements named.*

Proof. Assume first that G is definably connected. It follows that it is divisible.

By [22], there exists $G_0 \subseteq G$ a torsion-free definable subgroup of G with G/G_0 definably compact. Because G_0 is divisible and torsion-free it is elementarily equivalent to $\mathbb{R}^{\dim G_0}$. The group $G_1 = G/G_0$ is a definably compact, definably connected group and therefore by [9] (and, in the case that $\mathcal{M}$ expands an ordered group also by [10] and [17]), $Tor(G_1)$ is isomorphic to the torsion group of the real torus $\mathbb{T}^{\dim G_1}$. It follows (say, by 11.2) that G_1 is elementarily equivalent to the semialgebraic $\mathbb{T}^{\dim G_1}$ and G is elementarily equivalent to the group $\mathbb{R}^{\dim G_0} \times \mathbb{T}^{\dim G_1}$.

Finally, consider the divisible hull H in G of the group generated by the set C and $Tor(G)$. By 11.1, H can be written as the direct sum of $Tor(G)$ and $\mathbb{Q}c_1 \oplus \cdots \oplus \mathbb{Q}c_k$, for some $c_1, \ldots, c_k \in C$. By 11.2, H is an elementary subgroup of G. It can be realized also as an elementary subgroup of $\mathbb{R}^{\dim G_0} \times \mathbb{T}^{\dim G_1}$. Moreover, since all torsion elements are real algebraic and we can also choose real algebraic elements which are torsion-free and $\mathbb{Q}$-independent (using the well-known fact that the field of real algebraic numbers is infinite-dimensional as a $\mathbb{Q}$-vector space), it follows that $\langle G, +, \{g : g \in C \cup Tor(G)\}\rangle$ is elementarily equivalent to $\mathbb{R}^{\dim G_0} \times \mathbb{T}^{\dim G_1}$, with names for all torsion elements and finitely many other elements, all real algebraic.

Assume now that G is not definably connected. Then it equals a direct sum of its connected component G^0 and a finite group and therefore, by the above, it is elementarily equivalent to a semialgebraic group H of the same dimension, which can be defined over the real algebraic numbers. We can handle similarly finitely many named elements in G. $\square$

Given an expansion $\mathbf{A}$ of an abelian group A, consider a sub-language L_{ab} which has a predicate R_S for every 0-definable (in $\mathbf{A}$) subgroup $S \subseteq A^n$, $n \in \mathbb{N}$, as well as symbols for $+$ and 0. Let $\mathbf{A}_{ab}$ denote the reduct of $\mathbf{A}$ to L_{ab}. Such a structure has been sometimes called an *abelian structure*. We call the subgroups S above the

basic ones in the structure $\mathbf{A}_{ab}$. If B is a subgroup of A then we denote by $\mathbf{B}_{ab}$ the L_{ab}-induced structure on B, namely the interpretation of R_S is just its intersection with B^n. The next fact is a restatement of 7.2, but we add a bit more information in item 1.

Fact 11.4. *In the above setting (no o-minimality is assumed)*

(1) *(i) The theory of the structure $\mathbf{A}_{ab}$ eliminates quantifiers (in the language L_{ab}).*
(ii) Moreover $Th(\mathbf{A}_{ab})$ is axiomatized as follows: (a) Axioms for abelian groups, (b) Each symbol R_S denotes a subgroup, (c) Axioms for the defining properties of R_S: If S_1 is a projection of S_2 then R_{S_1} denotes the corresponding projection of R_{S_2}, and if $S_1 = \{x \in A^n : S_2(x, 0)\}$ then R_{S_1} denotes the corresponding fiber of R_{S_2}, (d) Axioms about the index (a given finite number or ∞) of R_{S_1} in R_{S_2} whenever $S_1 \leqslant S_2 \leqslant A^n$ are basic.

(2) *Assume that $B \leqslant A$ is a subgroup of A.*
Then $\mathbf{B}_{ab} \prec \mathbf{A}_{ab}$ if and only if the following hold:
(i) For every 0-definable (in $\mathbf{A}_{ab}$) subgroup $S \leqslant A^{n+k}$ and $b \in B^k$,
$$S(B^n, b) \neq \emptyset \Leftrightarrow S(A^n, b) \neq \emptyset.$$
(ii) For all 0-definable (in $\mathbf{A}_{ab}$) subgroups $S_1 \leqslant S_2 \leqslant A^n$,
$$[S_2 : S_1] = [S_2 \cap B^n : S_1 \cap B^n],$$
with the meaning that if this index is infinite on one side then it is infinite on the other.

(3) *Assume that $\mathbf{A}_{ab}$ has DCC on 0-definable subgroups. Then, for every $\mathbf{B}_{ab} \prec \mathbf{A}_{ab}$ there exists a surjective group homomorphism $\phi : A \to B$ which is the identity map on B and in addition sends every 0-definable $S \subseteq A^n$ onto $S \cap B^n$. (We call such a ϕ* a homomorphic retract*).*

Proof. (1)(i) is proved in [11].
1(ii) can be extracted from the proof of the quantifier elimination result in [11], in exactly the same way as the analogous statement for theories of modules is deduced from the proof of *pp* elimination for modules. See Theorem 1.1 and Corollary 1.5 of [25]. In fact in the statements on indices only basic subgroups of A itself (rather than A^n) need be considered.

(2) follows from (1).

(3) Using the quantifier elimination result above, the proof of (3) is basically identical to that of Theorem 2.8, p.28, in [24]. □

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